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**Optimization of Material Sourcing and Delivery Operations, and Assortment Planning for Vertically Differentiated Products and Bundles**

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# Optimization of Material Sourcing and Delivery Operations, and Assortment Planning for Vertically Differentiated Products and Bundles

**by**

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**DISSERTATION**

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

**DOCTOR OF PHILOSOPHY**

THE UNIVERSITY OF TEXAS AT AUSTIN

December 2009

Dedicated to my daughter Alice Xinyun Li and my husband Xiaolin Andy Li.

# Acknowledgments

I would like to express my sincere gratitude to my advisors Professors Anant Balakrishnan and Dorothee Honhon for their valuable guidance, insightful ideas, continuous encouragement and help on my study, research, career, and life. I feel so fortunate to have them as my supervisors.

I am deeply grateful to Professors Leon Lasdon, Steve Gilbert, and David Morton for being on my dissertation committee and giving their advice and guidance regarding my research and career. I'm also very thankful to Professor Qi Annabelle Feng for helpful research discussion, advice, and friendship.

I would like to thank Sree Jonnalagedda and Gang Li for helpful research discussion and sharing our life. I would also like to thank other friends in Austin for their friendship and help, which makes my study at Austin enjoyable and fruitful.

I am also grateful to the staff especially Hillary Patterson, Caroline Walls from Information Risk and Operation Management department for their assistance and support.

My special thanks go to my parents, Mr. Chaoping Pan and Ms. Xuejuan Yao, and other loving family members, for their unselfish love and support, and my husband, Dr. Xiaolin Li, for his advice, continuous encouragement, endless love, and spiritual support. Finally, I would like to express my greatest appreciation to my daughter, Alice Xinyun Li, for her understanding, love, and support beyond her age, who has been making my life so beautiful.

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Publication No. ________________

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The University of Texas at Austin, 2009

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Optimization of materials supply and inbound logistic operations has become increasingly important as firms have continued to pursue outsourcing options. Further, the proliferation of products and advances in information technology have greatly impacted retailers' marketing strategies in the past decade. In this dissertation, we address how to optimally develop integrated sourcing and delivery planning, and how to optimally offer vertically differentiated products and bundles. In the first essay, we address a combined sourcing and delivery planning optimization problem, which is motivated by a practical problem facing materials and supply planners for construction projects in a leading corporation. We develop a decision support model and an effective solution approach for integrated sourcing and delivery planning for bulk materials. This approach, implemented and currently in use at the company to support material delivery planning for track maintenance projects, has yielded significant savings of millions of dollars annually. In the second essay, we study the problem of a retailer managing a category of vertically differentiated products. We consider two settings: the exogenous prices case and the endogenous prices case. In the former case, the selling prices are exogenously determined and the retailer's only decision is to determine the set of products to offer. In the latter case, the retailer also determines the selling prices. We develop efficient methods to identify the optimal solutions for both cases and provide valuable insights and guidelines for practitioners. In the third essay, we study how to choose the optimal bundling strategy for a retailer offering vertically differentiated information goods. We characterize conditions under which pure bundling and mixed bundling strategies are optimal respectively. We provide efficient methods to identify which individual components to offer, whether or not to offer a bundle containing all the components and how to price the offered individual components and the bundle in order to maximize the retailer's profit.

# Table of Contents


**Acknowledgments** **v**

**Abstract** **vii**

**List of Tables** **xii**

**List of Figures** **xiii**

**Chapter 1. Introduction** **1**

**Chapter 2. Optimization of Integrated Souring and Delivery Planning for Bulk materials** **5**

2.1 Introduction . . . . . 5
2.2 Problem Description and Model Formulation . . . . . 10
2.2.1 Problem Context . . . . . 10
2.2.2 Model Formulation . . . . . 12
2.3 Strengthening the ISDP Model Formulation . . . . . 18
2.3.1 Delivery Consolidation Inequality . . . . . 18
2.3.2 Cumulative Deliveries Inequality . . . . . 20
2.3.3 Residual Capacity Inequality . . . . . 25
2.4 Methodology . . . . . 29
2.4.1 Preprocessing . . . . . 29
2.4.2 Separation Strategies . . . . . 30
2.4.3 LP-based Heuristic Procedure . . . . . 34
2.5 Implementation and Impact . . . . . 35
2.6 Conclusion . . . . . 41

**Chapter 3. Assortment Planning for Vertically Differentiated Products 43**

3.1 Introduction . . . . . 43
3.2 Literature Review . . . . . 47
3.3 The Exogenous Prices Case . . . . . 51
3.3.1 Model . . . . . 51
3.3.2 Results . . . . . 55
3.3.3 Example and Properties of the Optimal Solution . . . . . 59
3.4 The Endogenous Prices Case . . . . . 62
3.4.1 Model . . . . . 62
3.4.2 Results . . . . . 65
3.4.3 Example, Properties of the Optimal Solution and Special Cases . . . . . 69
3.5 Discussion and Insights . . . . . 76
3.6 Conclusion . . . . . 81

**Chapter 4. Optimal Pricing and Bundling of Vertically Differentiated Information Goods 84**

4.1 Introduction . . . . . 84
4.2 Literature Review . . . . . 85
4.3 Model . . . . . 88
4.4 Results . . . . . 92
4.5 Special Cases . . . . . 99
4.5.1 Same Positive Marginal Cost and Different Component Price 99
4.5.1.1 Example and insights . . . . . 100
4.5.2 Same Positive Marginal Cost and Same Component Price 104
4.5.3 Zero Marginal Cost and Different Component Price . . . . 106
4.5.4 Zero Marginal Cost and Same Component Price . . . . . 106
4.6 Conclusion . . . . . 108

**Chapter 5. Conclusion 111**

**Appendices 113**

**Appendix A. Supplemental Material for Chapter 2** **114**

**Appendix B. Supplemental Material for Chapter 3** **124**

**Appendix C. Supplemental Material for Chapter 4** **137**

**Bibliography** **147**

**Vita** **156**

# List of Tables


2.1 Performance of model. . . . . . . . . . . . . . . . . . . . . . . . 38

2.2 Solution characteristic of ISDP model. . . . . . . . . . . . . . . . 39

2.3 Comparison between solutions to ISDP model and myopic heuristics. 39

3.1 Notation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 53

3.2 Mapping between path and assortment for Example 3.1. . . . . . 60

3.3 Optimal assortments and optimal expected profit values as a function of $K$ and $b$ in Example 3.1. . . . . . . . . . . . . . . . . . . . 62

3.4 Mapping between paths and assortments in Example 3.2. . . . . 70

3.5 Optimal assortments, price vectors, and expected profit values as a function of $K$ and $b$ in Example 3.2. . . . . . . . . . . . . . . . 71

3.6 Solution methods to identify the optimal assortment. . . . . . . . 77

3.7 Comparison of the properties of the optimal assortment in exogenous and endogenous prices case. . . . . . . . . . . . . . . . . . . 79

4.1 Optimal $\theta$ and prices in Example 4.1. . . . . . . . . . . . . . . . . 98

# List of Figures


2.1 Event sequence diagram . . . . . . . . . . . . . . . . . . . . . . . . 13
2.2 LP solution of Example 1 . . . . . . . . . . . . . . . . . . . . . . . 20
2.3 IP solution of Example 1 . . . . . . . . . . . . . . . . . . . . . . . . 20
2.4 LP solution of Example 2 . . . . . . . . . . . . . . . . . . . . . . . 22
2.5 IP solution of Example 2 . . . . . . . . . . . . . . . . . . . . . . . . 22
2.6 Number of vehicles in a subsystem . . . . . . . . . . . . . . . . . . 23
2.7 LP solution of examples 3 for residual capacity inequality . . . . . 27
2.8 IP solution of examples 3 for residual capacity inequality . . . . . 27

3.1 Purchase probabilities in an example with 3 products. . . . . . . . 55
3.2 Graph for Example 3.1 with $K = 0$ and $b = 1$. . . . . . . . . . . . 60
3.3 Graph for Example 3.2 with $K = 0$ and $b = 1$. . . . . . . . . . . . 70
3.4 Graphical interpretation for $S^*$ in Example 3.2 with $K = 0$ and $b = 1$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 74
3.5 Upper envelope graph for Example 3.3. . . . . . . . . . . . . . . . . 76

# Chapter 1

# Introduction

Optimization of materials supply and inbound logistic operations has become increasingly important as firms have continued to pursue outsourcing options. In Chapter 1, motivated by a practical problem facing procurement and logistics planners at a leading U.S. company, we develop an optimization model to address an integrated sourcing and delivery planning problem. Given full-load customer demands, with delivery time window restrictions, and available sources of supply, the problem entails selecting the supplier(s) to use for each customer, assigning vehicle types to deliveries, and sequencing the materials pickup and delivery operations to meet all the delivery requirements on time at minimum total sourcing and delivery cost. Sequentially solving the sourcing and delivery problems can lead to solutions with high costs or even infeasible solutions due to supplier capacity constraints, customer delivery time windows, and limited number of vehicles. Researchers have worked on sourcing and material delivery problem separately; however, little attention has been paid to the combined sourcing and delivery problem. We formulate this problem as an integer programming model. To effectively solve the integrated sourcing and delivery planning problem, we develop several families of valid inequalities to enhance the model and propose a linear programming-based heuristic. When applied real problem instances from the company, our solution method provides near-optimal sourcing and delivery schedules within reasonable computational time. Estimated savings from using our integrated modeling approach exceeds three million dollars per year. This optimization model is a valuable decision support tool for tactical planning, capable of performing what-if analysis to provide useful managerial insights.

The proliferation of products have greatly impacted retailers' marketing strategies. A typical retailer carries dozens to hundreds of products in a popular product category. Advances in information technology make it possible for online retailers (E-tailers) to offer thousands of products in a category due to the absence of shelf space restriction, the huge costs savings of warehouse and the absence of walk-in customers. For instance, Newegg (www.newegg.com) carries more than 400 different USB flash drives and 1400 different televisions. The actual number of products that Newegg can choose to include in their assortment is even greater since not every brand and model is offered. Given such a variety of products, what products should a retailer offer in order to maximize the profit and how much should the retailer to charge for each offered product? A naive way is to enumerate all possible combinations of products with different prices and identify the most profitable one. However, the complexity of such method increases exponentially with the number of products that the retailer can choose from. To develop an efficient algorithm to identify the optimal set of products to offer and optimally price them is an important and challenging problem for large retailers. In Chapter 3, we consider the problem of a retailer managing a category of vertically differentiated products, i.e., products that differ in their quality level. The retailer has to pay a fixed cost per product included in the assortment and a variable cost per product sold. Quality levels, fixed and variable costs are exogenously determined. Customers differ in their valuation of quality and, given an assortment of products, choose the one (if any) that maximizes their utility. First we consider a setting in which the selling prices are also fixed and the retailer's only decision is to determine the set of products to offer. We find that the optimal assortment(s) depend(s) on the distribution of customer valuations and might include dominated products, i.e., products which are less attractive than at least one other product, on every possible dimension. We develop an efficient algorithm to identify the optimal assortment(s). Second, we consider a setting in which the retailer also determines the selling prices. We show that, under some mild conditions, the optimal assortment(s) does(do) not include any dominated product and does not vary with the distribution of customer valuations. We develop several efficient algorithms to identify the optimal assortment and optimally price the products.

Bundling has become a pervasive pricing strategy for sellers to increase their market share. For instance, telecommunication companies offer internet, phone, and TV services separately, as well as the bundle of these services; customers can choose to purchase an airline ticket, book a hotel, and rent a car separately or purchase a package including all the above products; the iTunes store sells songs individually as well as albums containing multiple songs. How do these sellers decide prices for each individual product and the bundle(s) in order to maximize the profit? In Chapter 4, we address the assortment planning problem for vertically differentiated information goods with price bundling. Motivated by the successful business of iTunes, we consider a special mixed bundling strategy: offering individual products, called components, and a bundle containing all the components. Customers agree on the ranking of popularity of each component. Each customer gets a utility from a product that is increasing in the popularity level and decreasing in the price. Depending on the utility that a customer gets from components and the bundle, he either makes no purchase, purchases one component, purchases multiple components, or purchases the bundle. The retailer pays a variable cost for each component/bundle sold. We address the questions about what components to offer individually, whether to offer the bundle, how much to charge for each offered component and/or the bundle. We develop efficient methods to identify the optimal solution. Moreover, we consider some special cases where the variable cost is the same for each component and/or the selling price is the same for each component, and provide valuable insights. For instance, we show that for a new product, low bundling pricing strategy is desirable to penetrate the early stage market. Then, as the market matures, a higher bundling price is optimal while the total market share increases.

# Chapter 2

# Optimization of Integrated Souring and Delivery Planning for Bulk materials

## 2.1 Introduction

Increasing transportation costs and trends toward more outsourcing have led firms to emphasize optimization of their supply and inbound logistics operations. Motivated by a practical problem facing materials supply planners for construction projects in a leading corporation, the paper addresses a joint sourcing and delivery planning optimization problem. In this problem, a set of geographically dispersed "customers" or projects require materials in bulk, i.e., in full vehicle loads, by specified due dates. We can select from among alternative suppliers for each customer. These suppliers vary in their prices, production capacities, and locations (and hence distances from customers). The planning problem requires selecting the suppliers to use for each customer, and sequencing the material pickup and delivery operations using available vehicles in order to meet the delivery requirements at minimum total procurement and delivery cost. Sequentially solving the sourcing and delivery problems can lead to solutions with high costs or even infeasible solutions due to supplier capacity constraints, customer delivery time windows, and limited number of vehicles. In the literature, researchers have worked on sourcing and material delivery problems separately, but little attention has been paid to the integrated sourcing and delivery problem (ISDP). The goal of this paper is to develop a decision support model and an effective solution approach for the ISDP. This approach, implemented and currently in use at the company to support material delivery planning activities, has yielded significant savings of over a million dollars annually.

Each year the company selects and plans the projects to be completed in the coming year. These projects require one or more full-loads of materials that can be supplied by alternative suppliers, and the materials must be delivered to the project sites before projects start. The previous planning process was manual, and could result in delayed deliveries and high procurement and transportation costs, which motivated us to address the ISDP that we describe next.

The ISDP entails three sets of interrelated decisions: sourcing, vehicle assignment, and routing. The sourcing decision requires selecting one or more suppliers from among the candidate suppliers for each customer. The prices offered by suppliers vary, as do the supplier-to-customer distances. Moreover, suppliers differ in the amount of material they can supply. These restrictions stem from limited weekly production and loading capacities, as well as limitations on the total amount of material that a supplier can provide over the planning horizon. We consider multiple vehicle types that differ in the time they require to pickup and deliver each load of material. The vehicle assignment decision entails selecting the vehicle type(s) that will serve each customer. Finally, the routing and scheduling decision requires sequencing the pickup and delivery activities for each vehicle so as to deliver on time to each customer while meeting the supply capacity restrictions. We aim to minimize the total cost of supplying all customers on time. This cost includes: (i) material procurement cost, which depends on the amount purchased from each supplier; (ii) transportation cost, which depends on the travel distance as well as the amount of the load; and, (iii) vehicle assignment cost, which varies by vehicle type and customer. Thus, the integrated sourcing and delivery problem seeks the supplier selection, vehicle assignment, and delivery sequencing plan that minimizes the total procurement, transportation, and vehicle assignment cost. The main purpose of the model is to support tactical planning of procurement and vehicle deployment during the planning horizon, for instance, six months to a year. The model can help address questions such as the following: do we have adequate cost-effective sources of supply close to customers, do we have adequate vehicles, can we meet all the deliveries on time, and how do we deploy the vehicles?

In the literature, several studies have addressed sourcing issues. Murthy *et al.* (2004) review the literature on optimization models for sourcing planning. Ustun & Demirtas (2008) provide a survey of optimization models for supplier selection problems with multi-criteria. The existing literature on sourcing focus on minimizing the purchasing cost and/or associated operational cost subject to different kinds of constraints. The constraints include satisfying the buyer's requirement for product quality and service level, and/or supplier capacity constraint, etc. Some models consider quantity discounts. To our best knowledge, no study has considered sourcing and delivery planning simultaneously as the ISDP does.

The Vehicle Routing Problem with Time Windows (VRPTW) is a core model to reduce delivery costs. This problem has been well studied in the literature. The VRPTW focuses on designing minimum cost multi-stop delivery routes for vehicles based at a single depot, given the locations, demands, and delivery time windows of customers. Researchers have developed numerous optimization methods and heuristics for this problem. Kallehauge *et al.* (2005) and Cordeau *et al.* (2007) provide a comprehensive survey of literature on the VRPTW. The ISDP differs from the VRPTW in the following ways. Typically, VRPTW models consider identical vehicles, with each vehicle starting and ending at the same depot. Accordingly, these models do not include sourcing and supplier assignment decisions. Dondo & Cerd (2007) address the multi-depot VRPTW in which each vehicle is housed in one of the depots, and customers can be supplied by any depot with no capacity constraints and no difference in prices of materials. For this problem, the authors propose a three-phase hierarchical hybrid approach combining a heuristic clustering algorithm with an optimization framework. Unlike the ISDP, for all the VRPTW problems studied in the literature, each customer's requirement is less than a full load, and so vehicles can serve multiple customers on each trip.

Bianco *et al.* (1995), Desaulniers *et al.* (1998), and Arunapuram *et al.* (2003) sstudy variants of vehicle routing problems with full-loads and time windows. Specifically, they consider the problem of scheduling a fleet of vehicles located at several depots to perform a set of tasks within specified time windows at minimum cost. Each task requires transporting one or more full-loads of material from one location to another location. These models do not include sourcing decisions, and only minimize the empty-load transportation cost. The papers develop different models for this problem, and propose branch-and-bound solution algorithms using column generation schemes.

As discussed above, past work has addressed sourcing problems and routing problems with time windows. No paper has worked on the joint sourcing and routing problem with full-loads and time windows. The ISDP model we develop and solve in this essay seeks to fill this gap. In particular, the ISDP decides which suppliers to assign to each customer, which period(s) to serve the customer based on due dates and supplier capacities, which vehicle to use for each delivery, and how to route the vehicles to satisfy these supplier assignment and timing choices. We must consider these decisions simultaneously in order to obtain an optimal sourcing and delivery plan.

We model the ISDP as a large-scale optimization problem which is computationally intractable (NP-hard). So, we focus on developing polyhedral approaches to generate near-optimal solutions within reasonable computation time. For this purpose, we identify several sets of inequalities based on the characteristics of fractional solutions to strengthen the model's linear programming (LP) relaxation. The strengthened model (the base model with the added inequalities) reduces the computational effort for exact procedures such as branch-and-bound, and improves the quality of heuristics solutions. The application of our approach to actual sourcing and delivery planning problems demonstrates that our solution methodology is quite effective. Furthermore, we obtain annual cost savings of over 10% on average compared to the total cost using a heuristic approach that the company developed. The results show that our model provides significant savings on the expensive construction projects. Moreover, the model developed in this paper has wide applications in industry, for instance, transporting corn from farms to ethanol refineries, shipping gas from oil refineries to distribution centers, supplying wood for furniture makers or building sites.

The remainder of this paper is organized as follows. Section 2.2 defines the ISDP formally and presents its mathematical programming formulation. Section 2.3 studies characteristics of the LP solution and develops various classes of valid inequalities to eliminate fractional LP solutions. Section 2.4 describes our solution method, including the preprocessing rules, the cutting plane procedure that we use to add valid inequalities, and the LP-based heuristic. Section 2.5 presents the implementation and impact of the ISDP model, and Section 2.6 concludes our work.

## 2.2 Problem Description and Model Formulation

### 2.2.1 Problem Context

The ISDP is motivated by the annual material supply planning task for construction projects. After the geographically-dispersed construction jobs are scheduled by specified dates, the corresponding materials, available from different suppliers, must be delivered to the job sites using a limited number of vehicles before the jobs start. Suppliers differ in their prices and supply time windows.

Some suppliers are open all the year round while others are open only in certain periods; for instance, suppliers in northern states cannot supply material in winter due to the severe weather. In addition, each supplier has a weekly production capacity and also an annual supply capacity. Vehicles of different types differ in the time spent on loading material at the supplier and unloading at the customer sites. Unloading at the customer site incurs additional costs that we capture using vehicle assignment costs. Hence, for each customer, the material can be delivered by vehicles of different types but with different assignment costs. This cost depends on the type of vehicles assigned to each customer. The ISDP considers which suppliers to serve for each customer, which vehicle type to use for each customer, and how to route the available vehicles to meet delivery requirements. These decisions together minimize the total cost of supplying all customers on time during the planning horizon. The cost includes: material procurement cost, which depends on the amount purchased from each supplier; transportation cost, which depends on the distance from the supplier to each customer, the return distance from each customer to the next supplier, and the per-mile equipment and consumables cost for the loaded and empty vehicle; and vehicle assignment cost, which varies by vehicle type and customer.

To model the integrated sourcing and delivery decisions, we exploit the characteristics of the company construction application context. In this setting, each customer (or job) requires one or more full loads or nearly full-loads deliveries, and so the vehicle can only deliver to one customer per trip. The trip consists of going to the supplier location, loading the vehicle, traveling to the customer site, unloading the vehicle, and then traveling to the next supplier (assigned to the next customer that the vehicle needs to serve). The planners assume that the forward trip time (to be loaded at a supplier, travel to a customer site, to be unloaded at the customer) is an integer multiple of the base period, as is a return trip time (travel to the next customer site). These problem features permit us to develop a time indexed model for the ISDP as described next.

### 2.2.2 Model Formulation

Let $J$ be the set of customers, $Q$ the set of suppliers, and $V$ the set of vehicle types. We index the time periods of the planning horizon from $t = 1$ to $n$, and define $T = 1, 2, , n$ as the set of base periods. Let $D_j$ denote the demand, in terms of number of deliveries (of full or nearly full-loads) required for customer $j$, to be delivered in the time window $TJ(j) \subseteq T$, with an average quantity of $B_j$ per delivery. For each customer $j \in J$, let $Q(j)$ denote the subset of suppliers that can supply customer $j$, and $R(j)$ the subset of suppliers to which vehicles can return after a delivery to this customer. These subsets may not include all suppliers because of travel distance restrictions and supplier availability. Each supplier $q \in Q$ can supply material in the time window $TQ(q) \subseteq T$. Let $I_q$ and $G_{qt}$ represent, respectively, supplier $q$'s initial inventory and production quantity in period $t$. In addition, this supplier can supply no more than $M_q$ units during the horizon. The set $J(q) \subseteq J$represents the subset of customers that supplier $q$ can serve. Let $N^v$ denote the number of available vehicles of type $v$. Vehicles can enter and leave the system at any period. Once a vehicle leaves the system, it can not re-enter. Vehicles of type $v$ take time units, an integer multiple of the base period, to load at supplier $q$, travel to customer $j$, and unload at customer $j$, and require periods to travel to the next supplier $q$. The events happen as follows. For each supplier $q$ in an operational period $t$, vehicles can enter the system at supplier $q$, or arrive at the supplier from customers at the beginning of period $t$, and then take time units to be loaded at supplier $q$, travel to customer $j \in J(q)$, and to be unloaded at this customer site. After unloaded at the beginning of period $t + \tau_{qj}^{v}$ , vehicles may leave the system, or take $\tau_{jq'}^{v}$ time units to travel to the next supplier $q' \in R(j)$(see Figure 2.1).

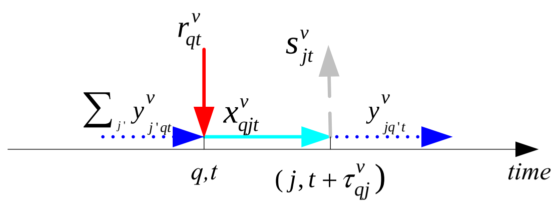


FIGURE 2.1: Event sequence diagram

We consider three types of costs. Let $F_{qj}$ and $E_{jq}$ denote the *transportation* cost per delivery from supplier $q$ to customer $j$ (full-load transportation cost) and from customer $j$ to supplier $q$ (empty-load transportation cost) respectively. The *procurement* price from supplier $q$ is $P_q$ per unit and the *vehicle assignment* cost at customer $j$ by a vehicle of type $v$ is $C_j^v$ per delivery.

To model the ISDP as an integer program, we define the following decision variables. For each customer $j \in J$, each supplier $q \in Q(j)$, each vehicle $v \in V$, and every period $t \in TQ(q)$ and $t + \tau_{qj}^{v} \in TJ(j)$ , the integer variable $x_{qjt}^{v}$ represents the number of vehicles of type $v$ that start to load material at supplier $q$

and travel to customer $j$ at the beginning of period $t$. This variable combines the supplier-to-customer assignment, vehicle-to-customer assignment, and delivery scheduling decisions. To model the return trips, let $y^v_{jqt}$ denote the number of vehicles of type $v$ that start to unload material at customer $j$ and travel to supplier $q$ at the beginning of period $t$, for all $j \in J$, $v \in V$, $q \in R(j)$, $t \in TJ(j)$ and $t + \tau^v_{jq} \in TQ(q)$. We define $r^v_{qt}$ as the number of vehicles of type $v$ that enter the system at supplier $q$ at the beginning of period $t \in TQ(q)$, and $s^v_{jt}$ as the number of vehicles of type $v$ that leave the system after delivering to customer $j$ at the beginning of period $t$, where $t-1 \in TJ(j)$. Using these integer decision variables, we can formulate the ISDP as the following optimization problem. Appendix A summarizes the notation used in the model. **[ISDP]**

Minimize

$$\begin{aligned} &\sum_v \sum_j \sum_q \sum_t P_q B_j x^v_{qjt} + \sum_v \sum_j \sum_q \sum_t (F_{qj} x^v_{qjt} + E_{jq} x^v_{jqt}) \\ &+ \sum_v \sum_j \sum_q \sum_t C^v_j x^v_{qjt} \end{aligned} \tag{2.1}$$

subject to

$$\sum_v \sum_j \sum_{t' \leq t} B_j x^v_{qjt'} \leq min\{I_q + \sum_{t' \leq t} G_{qt'}, M_q\} \text{ for all } t \in TQ(q), q \in Q, \quad (2.2)$$

$$\sum_q \sum_t r^v_{qt} \leq N^v \qquad \text{for all } v \in V, \quad (2.3)$$

$$\sum_v \sum_q \sum_t x^v_{qjt} = D_j \qquad \text{for all } j \in J, \quad (2.4)$$

$$\sum_q \sum_t r^v_{qt} + \sum_j y^v_{jq(t-\tau^v_{jq})} = \sum_j x^v_{qjt} \text{ for all } v \in V, q \in Q, t \in TQ(q), \quad (2.5)$$

$$\sum_q x^v qj(t - \tau^v_{qj}) = \sum_q y^v_{jqt} + s^v_{jt} \qquad \text{for all } v \in V, j \in J, t \in TJ(j), \quad (2.6)$$

$$x^v_{qjt} \geq 0, \text{ integer for all } v \in V, j \in J, q \in Q(j), t \in TQ(q), t + \tau^v_{qj} \in TJ(j), \quad (2.7a)$$

$$y^v_{jqt} \geq 0, \text{ integer for all } v \in V, j \in J, R \in R(j), t \in TJ(j), t + \tau^v_{jq} \in TQ(q), \quad (2.7b)$$

$$r^v_{qt} \geq 0, \text{ integer for all } v \in V, q \in Q, t \in TQ(q), \text{ and} \quad (2.7c)$$

$$s^v_{jt} \geq 0, \text{ integer for all } v \in V, j \in J, t - 1 \in TJ(j). \quad (2.7d)$$

The objective function (2.1) minimizes the total material purchasing, transportation, and vehicle assignment costs. Constraint (2.2) combines each supplier's production capacity constraint and total supply capacity during the planning horizon. This constraint specifies that, in every period t, the cumulative amount of material supplied by this supplier must not exceed the cumulative capacity (plus initial inventory) or the total supply capacity. Constraint (2.3) ensures that only the available vehicles can be used, while constraint (2.4) specifies that each customer's demand must be satisfied. Constraints (2.5)-(2.6) are flow conservation constraints at each customer and supplier location in every period during which they are operational. Constraints (2.5) specify that the inflow of vehicles to a supplier must equal the outflow (number of vehicles delivering material to customers) at the beginning of each period. The inflow is the number of vehicles entering the system plus the number of vehicles returning after delivering to customers at the beginning of the period. Constraint (2.6) states that, after each delivery, the vehicle must go to the next supplier or leave the system. Constraints (2.7a)- (2.7d) are the integrality requirements.

We refer to this formulation as a vehicle flow model since it considers the total number of vehicles arriving/leaving customer and supplier locations in each operational period rather than the routing of individual vehicles. From the vehicle flow solutions, we can easily obtain individual vehicle routes using a path decomposition method. Alternatively, we can formulate the problem using a vehicle indexed formulation that directly models the individual vehicle routes. However, this latter formulation vastly increases the number of variables and constraints. Moreover, this disaggregate vehicle-indexed formulation does not provide tighter linear-programming bounds. The larger problem size combined with the symmetry in the vehicle-indexed model can significantly increase computational time.

The ISDP model can easily incorporate user-specific requirements, such as: (a) pre-specified assignments of vehicle types to suppliers or customers; (b) delivery grouping, due to restrictions on the set of customers that must be served by the same vehicle type, and/or customers requiring consecutive deliveries; and,

(c) other costs and capacities, for instance, maximum number of loads per week at each supplier. Incorporating such requirements needs only adding some variables, constraints, and/or terms in the objective function without changing the underlying model structure.

A special case of the ISDP model is the single vehicle and single supplier problem with unlimited supply capacity, i.e., one vehicle makes deliveries to all customers supplied by one single source without any capacity constraint. This problem is polynomially solvable. Based on the demand of customers, we can identify the last delivery period $t'$ when the vehicle completes all deliveries and leaves the system. Among all the customers with due date later than period $t'-1$, we identify the customer $j$ with the largest distance between the supplier and the customer. The optimal solution makes the last delivery to customer $j$ in period $t'$ and to other customers in increasing order of due dates before period $t'$. Another special case of the ISDP model is the single vehicle and customers with unit demand problem, where each customer has a pre-specified unique source with unlimited supply capacity. In this case, the problem only involves determining the delivery sequence. We can polynomially transform the Traveling Salesman Problem (TSP) to this special case; therefore, the ISDP is NP-hard. To effectively solve this problem, we propose several classes of valid inequalities to strengthen the model's LP relaxation in the next section.

## 2.3 Strengthening the ISDP Model Formulation

To effectively solve the ISDP, we have developed several families of valid inequalities that strengthen the model formulation, thereby increasing the linear programming lower bound. During the past decade, researchers have successfully applied polyhedral approaches to solve many different classes of difficult integer programming problems; see, for instance, Bard *et al.* (2002) for the VRPTW. This experience has shown that adding valid inequalities not only reduces the enumeration effort for branch-and-bound and but can also help generate good heuristic solutions. Developing valid inequalities requires understanding the problem and solution structure to identify why and how the linear programming relaxation can reduce cost by selecting fractional values for the decision variables. For the ISDP, fractional solutions mainly stem from the LP relaxation's attempt to reduce the transportation cost. We illustrate how the LP relaxation of [ISDP] can achieve a lower optimal value than the IP value by examples before we develop valid inequalities. Next, we present the following three sets of valid inequalities to eliminate the fractional solutions: *delivery consolidation inequalities*, *cumulative deliveries inequalities*, and *residual delivery inequalities*. For notation simplification, we drop the index $v$ since all the inequalities apply for each vehicle type $v$.

### 2.3.1 Delivery Consolidation Inequality

Consider the following simple problem with two customers $j_1$ and $j_2$, two suppliers $A$ and $B$, and one vehicle. Each customer requires one full-load of material that can be delivered in any period of the first four periods, i.e., by the end of fourth period. Customer $j_1$ is close to supplier $A$ ($\tau_{Aj_1} = 1$ ) and far from supplier $B$ ($\tau_{Bj_1} = 2$ ), while customer $j_2$ is close to supplier $B$ ( $\tau_{Bj_2} = 1$ ) and far from supplier $A$ ($\tau_{Aj_2} = 2$ ). So, the transportation cost from customer $j_1$ to supplier $B$ is high, as is the transportation cost from customer $j_2$ to supplier $A$. To avoid these transportation costs, the linear programming relaxation schedules two "half" deliveries to each of the customers. In particular, it assigns half a vehicle to deliver material from supplier $A$ to customer $j_1$ in periods 1 and 3, and the other half vehicle to deliver material from supplier $B$ to customer $j_2$ during these two periods (see Figure 2.2). In contrast, the optimal solution requires first picking up material at supplier $A$ and delivering to customer $j_1$ in period 1, and then traveling to supplier $B$ (from customer $j_1$'s location) in the next two periods, picking up material, and delivering to customer $j_2$ in the fourth period (see Figure 2.3). To eliminate the fractional LP solution, we can impose the requirement that, in each period, a vehicle can travel from a supplier $q$ to a customer $j$ with unit demand only if the vehicle arrives at $q$ from another customer (other than customer $j$) or enters the system at the beginning of this period. Let $J_1$ denote the subset of customers with unit demand. Mathematically, we can express this requirement for customers with unit demand as follows.

$$x_{qjt} \leq \sum_{j' \neq j} y_{j'q(t-\tau_{j'q})} + r_{qt} \text{ for all} j \in J_1, q \in Q(j), t \in TQ(q). \tag{2.8}$$

We call constraint (2.8) the *delivery consolidation inequality.* Adding this constraints eliminates the previous LP solution and obtains the optimal IP solution for this example.

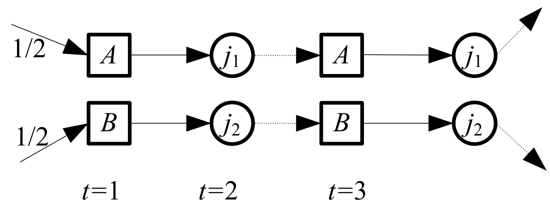


FIGURE 2.2: LP solution of Example 1

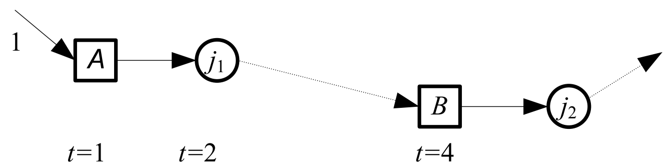


FIGURE 2.3: IP solution of Example 1

*Proposition* 2.3.1. The delivery inequality (2.8) is valid for the *ISDP* and tightens the model's LP relaxation.

### 2.3.2 Cumulative Deliveries Inequality

Consider another problem with four customers (customer $j_1$ to $j_4$), two suppliers $A$ and $B$, and one vehicle. Each customer requires one full-load of material by period 10. Customers $j_1$ and $j_2$ are close to supplier $A$ ($\tau_{Aj_1} = \tau_{Aj_2} = 1$ )and far from supplier $B$($\tau_{Bj_1} = \tau_{Bj_2} = 2$ ), while customers 3 and 4 are close to supplier $B$ ( $\tau_{Bj_3} = \tau_{Bj_4} = 1$ )and far from supplier $A$ ( $\tau_{Aj_3} = \tau_{Aj_4} = 2$). Correspondingly, the transportation cost from customer $j_1$ (customer $j_2$) to supplier $B$ is high, as is the transportation cost from customer $j_3$ (customer $j_4$) to supplier $A$. To save these high transportation costs, the linear programming relaxation assigns "half" $A$ vehicle to deliver material from supplier $A$ to customer $j_1$ and customer $j_2$ alternately and the other half vehicle to make deliveries to customer $j_3$ and customer $j_4$ from supplier $B$ alternately (see Figure 2.4). However, the optimal solution as shown in Figure 2.5 first makes deliveries from supplier $A$ to customers $j_1$ and $j_2$ in the first three delivery periods , and then travels to supplier $B$ (from customer $j_2$'s location) to deliver material to customers $j_3$ and $j_4$ in the remaining periods. The delivery consolidation inequality (2.8) cannot cut off the fractional solution. Observe that only half $A$ vehicle makes delivery to each customer while all customer demands are fully met. We use $J'$ to represent the customer subset consisting of customer $j_1$ and $j_2$. Let $A$ subsystem consist of customers in $J'$, supplier $A$, and time interval $[t_1, t_2] = [1, 7]$. Then, the number of vehicles in the subsystem is  while the proportion of satisfied demand is 1. We can eliminate this fractional solution by requiring that at least one vehicle devotes to each customer with fully met demand. The following *cumulative deliveries inequality* captures this restriction.

$$\sum_{t=t_1}^{t_2} r_{qt} + \sum_{j\in J'} \sum_{t=t_1-\tau_{qj}}^{min\{t_1-1, t_2-\tau_{qj}-\tau_{jq}\}} x_{qjt} + \sum_{q'\neq q} \sum_{j\in J'} \sum_{t=t_1-\tau_{q'j}}^{t_2-\tau_{q'j}-\tau_{jq}} x_{q'jt}$$
$$+ \sum_{j\in J'} \sum_{t=t_1-\tau_{jq}}^{min\{t_1-1, t_2-\tau_{jq}\}} y_{jqt} + \sum_{j\notin J'} \sum_{t=t_1-\tau_{jq}}^{t_2-\tau_{jq}} y_{jqt} \geq \sum_{t=t_1}^{t_1} x_{qj^*t}/D_{j^*}$$
$$\text{for all } q \in Q, t_1, t_2 \in TQ(q), t_1 + 1 \leq t_2, J' \subseteq J, j^* \in J' \qquad (2.9)$$

The left hand side (LHS) of inequality (2.9) represents the number of

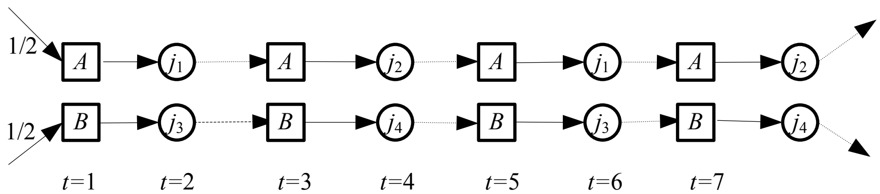


FIGURE 2.4: LP solution of Example 2

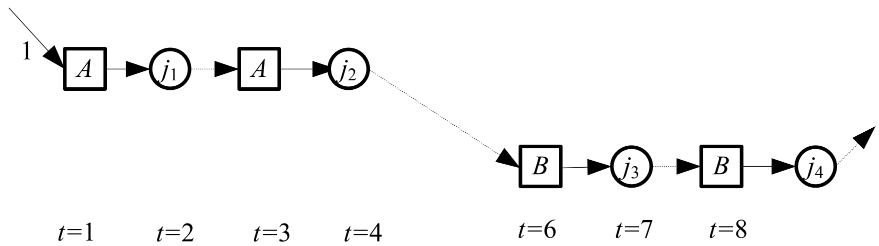


FIGURE 2.5: IP solution of Example 2

vehicles entering the subsystem at supplier $q$ and customer subset $J'$ as shown in Figure 2.6. The dot-dashed line represents the flow entering the subsystem at supplier $q$ from outside (the first term); the dashed line and the dotted lines indicate the flow entering the subsystem at customer subset $J'$ from supplier $q$ (the second term) and from other suppliers (the third term) respectively, the dot-dot-dashed line and the solid lines represent the flow entering the subsystem at supplier $q$ from customers in subset $J'$ (the fourth term) and customers not in $J'$ (the last term) respectively. We call these five terms flow from outside system, flow from inside supplier, flow from outside suppliers, flow from inside customers, and flow from outside customers. The LHS includes the vehicles in the subsystem that can make delivery from supplier $q$ in the time interval. So, we take into account the flow from insider supplier (the second term) not later than $t_1 - 1$ and $t_2 - \tau_{qj} - \tau_{jq}$, and the flow from inside customers (the fourth term) not later than $t_1 - 1$ and $t_2 - \tau_{jq}$. This inequality specifies that, for any subset of customers $J'$, supplier $q$, and time interval $[t_1, t_2]$, the number of vehicles entering this subsystem must equal or exceed the proportion of the total demand for each customer $j^* \in J'$that supplier $q$ serves in the time interval. The cumulative deliveries inequality is effective only for time intervals with at least one period since the constraint is implied by the flow conservation constraint (2.5) otherwise.

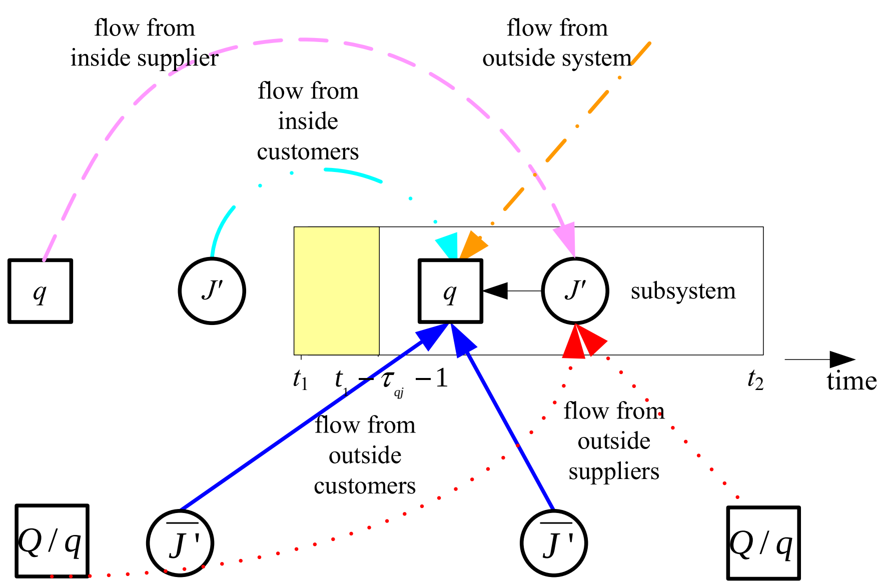


FIGURE 2.6: Number of vehicles in a subsystem

Notice that the second and third terms in the LHS of inequality (2.9) (the dashed line and dotted lines in Figure 2.6) represent the number of vehicles entering the subsystem at customers in $J'$. Actually, we only need to include these vehicles if they go to supplier $q$, and so we can strengthen this inequality by using the return trip variables ($\sum_{j \in J'} \sum_{t=t_1}^{t_1+\tau_{qj}-1} y_{jqt}$ ) in time interval $[t_1, t_1 + \tau_{qj} - 1]$ (shown in grey box in Figure 2.6). In addition, we do not need to include the number of vehicles entering at supplier $q$ and then immediately serving customers not in subset $J'$in period $t_1$, i.e., we can exclude $\sum_{j \notin J'} x_{qjt_1}$ from the last two terms in the LHS of inequality (2.9) representing the number of vehicles entering at supplier $q$. Therefore, we can strengthen the cumulative deliveries inequality as follows.

$$\sum_{t=t_1}^{t_2} r_{qt} + \sum_{j \in J'} \sum_{t=t_1}^{t_1+\tau_{qj}-1} y_{jqt} + \sum_{q' \neq q} \sum_{j \in J'} \sum_{t=t_1+\tau_{qj}-\tau_{q'j}}^{t_2-\tau_{q'j}-\tau_{jq}} x_{q'jt}$$
$$\sum_{j \in J'} \sum_{t=t_1-\tau_{jq}}^{min(t_1-1,t_2-\tau_{jq})} y_{jqt} + \sum_{j \notin J'} \sum_{t=t_1-\tau_{jq}}^{t_2-\tau_{jq}} y_{jqt} - \sum_{j \notin J'} x_{qjt_1} \geq \frac{\sum_{t=t_1}^{t_2} x_{qj^*t}}{D_{j^*}}$$
$$\text{for all } q \in Q, t_1, t_2 \in TQ(q), t_1 + 1 \leq t_2, J' \subseteq J, j^* \in J$$

Furthermore, we can generalize the cumulative deliveries inequality for a subset of suppliers $Q'$, i.e., for a subset of supplier $Q'$, a subset of customers $J'$, and time interval $[t_1, t_2]$, the number of vehicles entering this subsystem from outside must equal or exceed the proportion of the total demand for each customer $j^* \in J'$ that the subset of suppliers $Q'$ serve in the time interval. Let $\tau_{jQ'} = min_{q \in Q'} \tau_{jq}$ denote the shortest return trip time from customer $j$ to among any supplier $q \in Q'$, and $\tau_{Q'j} = min_{q \in Q'} \tau_{qj}$ denote the shortest forward trip time from among any supplier $q \in Q'$ to customer $j$. We use $z$ to represent the total number of vehicles in the subsystem, i.e,

$$
\begin{aligned}
z \;=\; & \sum_{q\in Q'}\sum_{t=t_1}^{t_2} r_{qt} + \sum_{j\in J'}\sum_{q\in Q'}\sum_{t=t_1}^{t_1+\tau_{qj}-1} y_{jqt} + \sum_{q\in Q'}\sum_{j\in J'}\sum_{t=t_1+\tau_{Q'j}-\tau_{qj}}^{min(t_1-1,t_2-\tau_{qj}-\tau_{jQ'})} x_{qjt} \\
& + \sum_{q\notin Q'}\sum_{j\in J'}\sum_{t=t_1+\tau_{Q'j}-\tau_{qj}}^{t_2-\tau_{qj}-\tau_{jQ'}} x_{qjt} + \sum_{j\in J'}\sum_{q\in Q'}\sum_{t=t_1-\tau_{jq}}^{min(t_1-1,t_2-\tau_{jq})} y_{jqt} \\
& + \sum_{j\notin J'}\sum_{q\in Q'}\sum_{t=t_1-\tau_{jq}}^{t_2-\tau_{jq}} y_{jqt} - \sum_{q\in Q'}\sum_{j\notin J'} x_{qjt_1}
\end{aligned}
$$

and the general version is

$$
z \geq \frac{\sum_{t=t_1}^{t_2} x_{qj^*t}}{D_{j^*}} \text{ for all } Q' \subseteq Q, t_1, t_2 \in TQ(q), t_1+1 \leq t_2, J' \subseteq J, j^* \in J \quad (2.10)
$$

*Proposition* 2.3.2. The cumulative deliveries inequality (2.10) is valid for the ISDP and tightens the model's LP relaxation.

### 2.3.3 Residual Capacity Inequality

Observe that the cumulative deliveries inequality is effective only when the number of vehicles in the system is less than one since the RHS of constraint (2.10) is at most one. Consider an example with two customers $j_1$ and $j_2$, two suppliers $A$ and $B$, and three vehicles. Customer $j_1$ demands three full loads of material by period 3, and customer $j_2$ requires three full-loads of material due in period 4. Customer $j_1$ is close to supplier $A$ ($\tau_{Aj_1} = 1$ ) and far from supplier $B$ ($\tau_{Bj_1} = 2$ ), while customer $j_2$ is close to supplier $B$ ( $\tau_{Bj_2} = 1$) and far from supplier $A$ ( $\tau_{Aj_2} = 2$). To avoid the high transportation cost from customer $j_1$ to supplier B, the linear programming relaxation assigns 3/2 vehicles to make deliveries to customer $j_1$ from supplier $A$, and the other 3/2 vehicles to deliver materials from supplier $B$ to customer $j_2$ (see Figure 2.7). In contrast, the optimal solution shown in Figure 2.8 requires that one vehicle continuously makes deliveries to customer $j_2$ from supplier $B$ and one vehicle delivers to customer $j_1$ from supplier $A$ in the first three periods, and the other one vehicle makes delivery to customer $j_1$ from supplier $A$ in the first period, and then travels to supplier $B$, picks up materials, and delivers to customer $j_2$ in the remaining periods. The cumulative deliveries inequality cannot eliminate this fractional solution. The residual capacity inequality developed next can cut off this fractional solution. Before presenting the residual capacity inequality, we introduce some additional notation. For any time interval $[t_1, t_2]$, customer subsets $J'$ and $J''$ with $J'' \subseteq J'$, and supplier subset $Q'$, let $W$ denote the total demand of customers in subset $J$" that can be satisfied in time interval $[t_1, t_2]$. We compute $W$ by summing up the demands of all customers in subset $J$" whose due date is later than period $t_1 - 1$. Let $\tau_{Q'J''} = min_{j \in J''} \tau_{Q'j} + min_{j \in J''} \tau_{jQ'}$ denote the shortest trip time among suppliers in $Q'$ and customers in $J''$, and $U = \lfloor (t_2 - t_1 + 1)/\tau_{Q'J''} \rfloor$ denote the maximum number of deliveries that a vehicle can make to customer subset $J''$ from supplier subset $Q'$ in the time interval. Define $K = \lceil W/U \rceil$ as the minimum number of vehicles needed to satisfy demand $W$, $\lambda = KU - W$ as the excess capacity, and $R = (U - \lambda)$ as the residual capacity. The residual capacity inequality requires that, the last vehicle must make at least $R$ deliveries if the number of vehicles serving the system is less than the minimum number of vehicles needed to satisfy the demand. Let $z$ denote the number of vehicles in the subsystem of supplier subset $Q'$, customer subset $J'$ in time interval $[t_1, t_2]$. Mathematically, we can express the *residual capacity inequality* as follows.

$$\sum_{q \in Q'} \sum_{j \in J''} \sum_{t=t_1}^{t_2} x_{qjt} + R(K - Z) \leq W$$
$$\text{for all } Q' \subseteq Q, t_1, t_2 \in TQ(q), t_1 + 1 \leq t_2, J' \subseteq J, J'' \subseteq J' \qquad (2.11)$$

In the example shown in Figure 2.7, let time interval $[t_1, t_2] = [1, 3]$, customer subset $J' = j_1, J'' = j_1$, supplier subset $Q' = A$. We can compute $U = 2$, $W = 3$, $K = 2$, and $R = 1$. The LHS $(= 7/2)$ of constraint (2.11) is greater than the RHS $(= 3)$, so adding this constraint eliminates this fractional solution.

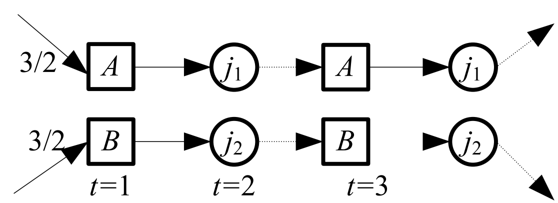


FIGURE 2.7: LP solution of examples 3 for residual capacity inequality

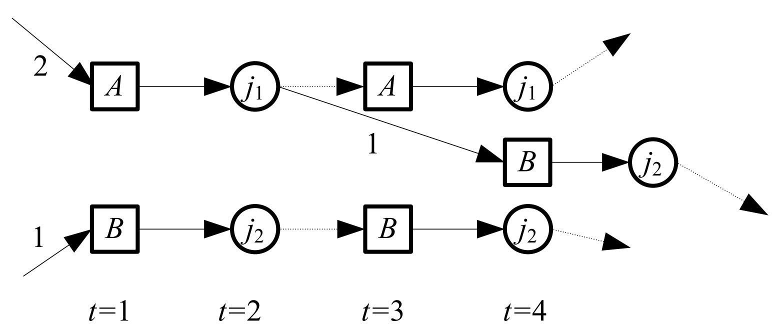


FIGURE 2.8: IP solution of examples 3 for residual capacity inequality

For a vehicle entering the subsystem from customer $j \notin J'$ or supplier $q \notin$

$Q'$ at time $t$, the residual capacity $R'(t)$of this vehicle is $min(R, \lfloor (t_2 - t)/\tau_{Q'J'} \rfloor)$. For a vehicle leaving the subsystem through customer $j \in J'$ or supplier $q \in Q'$ at time $t$, we can add terms to compensate the possible loss of vehicle capacity $R''(t) = max\{0, R - \lfloor (t + 1 - t_1)/\tau_{Q'J'} \rfloor\}$. Therefore, we can further strengthen the inequality by accounting for late arriving and early exiting vehicles within the time interval. Let

$$
\begin{aligned}
z' \;=\; & \sum_{q \in Q'} \sum_{t=t_1}^{t=t_2} R'(t) r_{qt} + \sum_{j \in J'} \sum_{q \in Q'} \sum_{t=t_1-\tau_{jq}}^{min(t_1-1, t_2-\tau_{jq})} R'(t+\tau_{jq}) y_{jqt} \\
& + \sum_{j \notin J'} \sum_{q \in Q'} \sum_{t=t_1-\tau_{jq}}^{t_2-\tau_{jq}} R'(t+\tau_{jq}) y_{jqt} \\
& + \sum_{q \in Q'} \sum_{j \in J'} \sum_{t=t_1-\tau_{qj}}^{min(t_1-1, t_2-\tau_{qj}-\tau_{jQ'})} R'(t+\tau_{qj}+\tau_{jQ'}) x_{qjt} \\
& + \sum_{q \notin Q'} \sum_{j \in J'} \sum_{t=t_1-\tau_{qj}}^{t_2-\tau_{qj}-\tau_{jQ'}} R'(t+\tau_{qj}+\tau_{jQ'}) x_{qjt} - \sum_{j \in J'} \sum_{t=t_1}^{t_2-\tau_{jQ'}} R''(t) s_{jt} \\
& - \sum_{j \in J'} \sum_{q \notin Q'} \sum_{t=t_1}^{t_2-\tau_{jQ'}} R''(t) y_{jqt} - \sum_{q \notin Q'} \sum_{j \notin J'} \sum_{t=t_1}^{t_2-\tau_{jQ'}} R''(t) x_{qjt},
\end{aligned}
$$

then the strengthened version is

$$
\sum_{q \in Q'} \sum_{j \in J''} \sum_{t=t_1}^{t_2} x_{qjt} + RK - z' \le W
$$
$$
\text{for all } Q' \subseteq Q, t_1, t_2 \in TQ(q), t_1 + 1 \le t_2, J' \subseteq J, J'' \subseteq J' \tag{2.12}
$$

*Proposition* 2.3.3. The residual capacity inequality (2.12) is valid for the ISDP and tightens the models LP relaxation.

In summary, motivated by examples shown above, we developed three classes of valid inequalities *delivery consolidation inequality*, *cumulative deliveries inequality*, and *residual capacity inequality* to tighten the base model. The first class of inequality includes one constraint for each choice of customer $j \in J_1$, supplier $q \in Q(j)$, period $t \in TQ(q)$, while the other two classes of inequalities each contain one constraint for subsets of customers, suppliers, and time interval. Since there are exponential numbers of such choices, we add violated cumulative delivery and residual capacity inequalities via a cutting plane procedure using heuristic separation methods, as discussed next.

## 2.4 Methodology

To solve the ISDP effectively, we first preprocess problems by eliminating some variables to reduce the problem size. Then, we use a cutting plane method to iteratively add violated inequalities described in Section 2.3 in order to tighten the base model. After we obtain the strong formulation (base model with additional valid inequalities), our solution method applies a tailored LP-based heuristic to generate good solutions to speed up the branch-and-bound procedure.

### 2.4.1 Preprocessing

Preprocessing can reduce the problem size, thereby obtaining the solution more quickly. We preprocess problems by eliminating variables as follows.

- *Elimination of vehicle exit variables.* Recall that $\tau_{QJ}$ denote the shortest trip time among suppliers in $Q$ and customers in $J$. Compute the available cumulative vehicle delivery capacity ($N'_t = N\lfloor t/\tau_{QJ}\rfloor$) and the cumulative customer demand ($D'_t = \sum_{TJ(j)\subseteq\{1,\ldots,t\}} D_j$) by each period $t$. Identify the period $\bar{t}$ when the difference between the cumulative vehicle delivery capacity and customer demand ($N'_t - D'_t$)is the smallest. We can infer the earliest period $t_e$ that any vehicle can leave the system by checking . Before period $t_e$, we eliminate all vehicle exit variables ($s$ variables), thereby tightening the models LP relaxation.

- *Elimination of delivery variables.* Compute the available cumulative vehicle delivery capacity ($N'_t = N\lfloor t/\tau_{QJ}\rfloor$) and the cumulative customer demand ($D'_t = \sum_{TJ(j)\subseteq\{1,\ldots,t\}} D_j$) by each period $t$. Identify the period $\bar{t}$ when the difference between the cumulative vehicle delivery capacity and customer demand ($N'_t - D'_t$)is the smallest. If $(N'_{\bar{t}} - D'_{\bar{t}}) = 0$, we eliminate all delivery variables ($x$, $y$, $s$ variables) before period $\bar{t}$.

### 2.4.2 Separation Strategies

The classes of inequalities all entail adding many additional constraints to the model. The delivery consolidation inequality requires adding a constraint for each combination of each customer with unit demand, supplier and time period; the other two classes of inequalities require identifying the subsets of customers, suppliers, and time intervals. So, adding all of the inequalities a priori to the base model can vastly increase the problem size and computational time. Our cutting plane method starts with solving the LP relaxation of the base model, and then iteratively adds the violated inequalities of each class separately to strengthen the model. Specifically, set the violation tolerance range $[\bar{\alpha}, \underline{\alpha}]$, and applies the following procedures for residual capacity inequalities, cumulative deliveries inequalities, and delivery consolidation inequalities respectively.

- Step 1. Set violation tolerance $\alpha = \bar{\alpha}$ .
- Step 2. Add the violated residual capacity inequalities for the violation tolerance value $\alpha$ to the current formulation, and re-solves the LP relaxation until no violated inequalities can be identified or the LP relaxation value does not increase so much.
- Step 3. Set $\alpha = \alpha/2$ and repeat step 2 until $\alpha \leq \underline{\alpha}$.

The key step in the cutting plane algorithm is solving the separation problem to identify the violated inequalities of each class. For the class of delivery consolidation inequality, we avoid checking each combination of customer with unit demand, supplier and time period by adopting the separation strategy described below in order to identify the violated inequality more efficiently. The other two classes of inequalities require specifying customer subsets, a supplier subset and time interval for each constraint. To find all the violated inequalities or the most violated inequality is computationally intractable, so we develop heuristic approaches to identify the subsets leading to constraints with large violations.

For a customer $j$ with unit demand and supplier $q \in Q(j)$, the incumbent LP fractional solution might violate the *Delivery Consolidation Inequality* only when the supplier serves the customer in consecutive periods. So, we check whether the delivery consolidation inequality is violated at period $t$ for customer $j$ and supplier $q$ only if the LP relaxation solution of the base model has vehicle flow from $q$ to $j$ in both period $t$ and $t-1$.

Since both classes of *Cumulative Deliveries Inequality* and *Residual Capacity Inequality* require selecting a subsystem consisting of a customer subset, a supplier subset, and time interval for each constraint, we use the following common procedure to construct the subsystem. For each customer $j^*$, we construct a subsystem consisting of customer subset $J'$, supplier subset $Q'$, and time interval with the largest violation by the incumbent LP solution. So, for each customer $j^* \in J$, we initialize the customer subset $J'$ by including $j^*$.

- Step 1. *Choose time intervals.* For each customer $j^* \in J$, identify the delivery time interval, i.e., the earliest delivery period and latest delivery period. Both inequalities are most likely effective for time intervals with high delivery flow to the customer. So, we sort the periods in the decreasing order of delivery flow, and extend the time interval $[t_1, t_2]$ by including the periods in the list one by one.

- Step 2. *Construct subsystem.* Construct the customer subset and supplier subset for each time interval as follows.

  – 2A. Include supplier $q$ serving customer $j^*$ at $t_2$ to $Q'$.

– 2B. For each period starting from $t_2 - 1$ to $t_1$, augment the customer subset and supplier subset by following the principle that the augmented subsets can make the LHS (the number of vehicles in the subsystem) smaller.

– 2C. For other customers and suppliers, we add one by one to the subsystem only if adding it would make the LHS smaller (the number of vehicles in the subsystem).

For the *cumulative deliveries inequality*, for each customer $j^*$, we check the violation value for each identified subsystem, and add the most violated constraint to the model. The *Residual Capacity Inequality* requires selecting a customer subset $J''$ to calculate the minimum number of vehicles $K$ needed to meet the customer demand (workload) in the time interval. Since the *Residual Capacity Inequality* is effective only when $K$ is more than $\lceil z \rceil - 1$ and less than $\lceil z \rceil$, we augment the customer subset $J''$ from $j^*$ until the total customer demand in $J''$ needs $K(\lceil z \rceil - 1 < K < \lceil z \rceil)$vehicles to make deliveries. In addition, to further limit the model size when adding the *Residual Capacity Inequalities*, for each customer subset $J''$ , we choose the time interval with the largest violation. Given the incumbent fractional solution $(\bar{x}_{qjt}, \bar{y}_{jqt}, \bar{r}_{qt}, \bar{s}_{jt})$ , the detailed separation procedure for the *Residual Capacity Inequality* is shown in Appendix A.

### 2.4.3 LP-based Heuristic Procedure

The LP solution to the strong model serves as a good starting point for generating heuristic solutions. We apply a LP-based approach to generate a heuristic solution to reduce the enumeration effort in the branch-and-bound procedure. Based on the fractional solution to the strong model (base model with added violated inequalities), we narrow the delivery window for each customer and restrict the number of suppliers that can serve a customer. We obtain a heuristic solution to the model with these additional constraints. More specifically, the LP based heuristic applies the following three steps from the fractional solution $(\bar{x}_{qjt}, \bar{y}_{jqt}, \bar{r}_{qt}, \bar{s}_{jt})$ to the strong model.

- Step 1. *Narrow customer delivery window.* For each customer $j$, identify the delivery time window $[\bar{t_1}, \bar{t_2}]$ in the LP solution, i.e., $\sum_{q \in Q} \sum_{t=1}^{\bar{t_1}-1} \bar{x}_{qjt} = 0, \sum_{q \in Q} \bar{x}_{qj\bar{t}_1} > 0, \sum_{q \in Q} \sum_{t=\bar{t}_1}^{\bar{t_2}} \bar{x}_{qjt} = D_j$; and add constraints specifying no deliveries to the customer beyond the time window $[\bar{t_1}, \bar{t_2}]$.

- Step 2. *Restrict the number of suppliers that can serve a customer.* For each customer $j$, check if a candidate supplier $q$ serves the customer in the delivery time window $[\bar{t_1}, \bar{t_2}]$ in the LP solution, i.e., $\sum_{q \in Q} \sum_{t=\bar{t}_1}^{\bar{t_2}} \bar{x}_{qjt} > 0$. If supplier $q$ does not serve customer $j$ in the delivery window, impose the restriction of no deliveries to the customer $j$ from supplier $q$ during the entire horizon.

If the above LP-based heuristic cannot find a feasible solution, we change the heuristic by relaxing the restriction on the starting delivery period of each customer in step 1. The result presented next demonstrates that our solution approach, combining model enhancements with the LP-based heuristic, is quite effective.

## 2.5 Implementation and Impact

The goal of this essay is to formulate the ISDP as an integer programming model, and develop valid inequalities and a LP-based heuristic to solve the problem effectively. We apply this approach to three sets of actual data from an actual company. Each data set has a limited number of available vehicles. Two data sets have homogenous vehicles while the other data set has heterogeneous vehicles. The suppliers have wide variation in price per ton and production capacities. Some suppliers are able to supply material for the entire horizon while others can supply material only in some time interval. Also, some suppliers have limits on the number of full loads per period, and total material availability during the entire horizon. Each customer prefers suppliers within pre-specified maximum distance from the customer site. If multiple suppliers meet distance restriction, we select those that are the cheapest in terms of procurement cost plus transportation cost, and if the cheapest suppliers have capacity restriction or are not open all the year-round, we add other candidate suppliers. Empty vehicles cannot travel more than pre-specified distance from customer site to the next suppler. Based on these criteria, for each customer, we create the list of candidate suppliers who can supply materials, and the list of suppliers that vehicles can go after unloading at the customer site.

We implement the model in Java programs using ILOG CPLEX 10.0 Concert Technology on a Linux server with 2 dualcore, hyperthreading 3.73 MHz Xeon processors and 24 GB of shared memory. We eliminate 6% of the variables through pre-processing. The problem size of base formulations for the available sets of data varies from around 63,000 variables and 9,000 constraints to 102,000 variables and 12,000 constraints. Adding valid inequalities reduces the enumeration effort for branch-and-bound and helps generate good heuristic solutions. To assess the effectiveness of the strengthened model and solution methodology, we apply a branch-and-bound procedure starting with the base model (base B&B approach) and compare the performance versus starting with the strong model and the heuristic solution (strong B&B approach). For both branch and-bound procedures, we run for one hour or to achieve 1% gap and 24 hours or to achieve 1% gap respectively, and terminate the procedure whichever occurs earlier. We assess the effectiveness of the strong model and associated heuristic by comparing the final gaps ((Best UB - Best LB)/Best UB %), final upper bounds, and computational times for the two approaches.

Table 2.1 summarizes the results using the base B&B and strong B&B approaches. As shown from Rows 1 to 6 in the table, the cutting plane procedure adds only a small number of cuts described in Section 2.3, around 10% of constraints, which limits the size of the strengthened model. The LP-based heuristic based on the LP solution from the strengthened model provides near-optimal solutions in short time without further applying branch-and-bound procedures. In contrast, the base branch and bound procedure (Rows 7 to 12) cannot solve the problem optimally at the gap of 1% within one hour. Row 5 shows the savings of the strong B&B approach over the base B&B approach. To achieve 1% gap, the base B&B approach needs much more time as shown in Row 6. The results demonstrate effectiveness of our solution methodology, valid inequalities with LP-based heuristic developed in Sections 2.3 and 2.4. Table 2.2 presents the solution characteristics of the ISDP model. 75% to 93% of deliveries are supplied by the cheapest suppliers (shown in Row 2), and 72% to 90% of deliveries are supplied by the closest suppliers (shown in Row 4).

To assess the benefit of using ISDP model in practice, we compare the solution obtained using a myopic heuristic with the optimal solution. The myopic heuristic starts with identifying the best supplier and the best customer to serve for each vehicle by considering the supplier-customer cost. This cost includes the transportation and purchasing cost from a candidate supplier $q$ to a candidate customer $j$, the vehicle assignment cost at customer $j$, and an incentive cost to induce an early delivery to customers with early due dates. For each period and each vehicle, the heuristic procedure checks if the vehicle completes the current forward trip in that period. When the forward trip is completed, the procedure chooses the next best supplier-customer pair by considering the above supplier customer cost plus the transportation cost from the current customer location to the candidate supplier q. During the process of choosing the best supplier-customer pair, the heuristic ensures the best candidate supplier has enough capacity. This heuristic procedure considers the sourcing, vehicle assignment, and routing decisions myopically instead of optimally. The ISDP model

| | | | Data set 1 | Data set 2 | Data set 3 |
|---|---|---|---|---|---|
| 1 | Strong B&B approach | No. of cuts added | 415 | 1170 | 886 |
| 2 | | No. of B&B nodes evaluated | 0 | 0 | 0 |
| 3 | | Final gap | 0.82% | 0.99% | 0.70% |
| 4 | | CPU time (mins) | 29 | 52 | 30 |
| 5 | | Cost savings over base model | 0.69% | 2.23% | 0.91% |
| 6 | | Cost time savings over base model (mins) | 360 | 1,388 | 233 |
| 7 | Base B&B approach (1% gap or 1 hour termination) | No. of B&B nodes evaluated | 4,346 | 1,042 | 4,799 |
| 8 | | Final gap | 1.77% | 3.35% | 1.61% |
| 9 | | CPU time (mins) | 60 | 60 | 60 |
| 10 | Base B&B approach (10% gap or 24 hour termination ) | No. of B&B nodes evaluated | 52,000 | 26,022 | 21,729 |
| 11 | | Final gap | 0.99% | 1.69% | 0.99% |
| 12 | | CPU time (mins) | 389 | 1,440 | 263 |

Table 2.1: Performance of model.

provides annual cost savings of more than 10% on average over the heuristic (see Table 2.3). The company previously developed the material procurement and delivery plan manually by choosing the closest suppliers. Sometimes, the demand of customers could not be met by due date, resulting in the delay of

| | | Data set 1 | Data set 2 | Data set 3 |
|---|---|---|---|---|
| 1 | No. of deliveries supplied by cheapest suppliers | 237 | 223 | 256 |
| 2 | % of deliveries supplied by cheapest suppliers | 75% | 86% | 93% |
| 3 | No. of deliveries supplied by closest suppliers | 226 | 219 | 247 |
| 4 | % of deliveries supplied by closest suppliers | 72% | 85% | 90% |

Table 2.2: Solution characteristic of ISDP model.

the following scheduled tasks. Taking this part of delay cost into account, the ISDP model saves more cost. The results also show that the transportation cost is around twice the procurement cost, the vehicle assignment cost is close to the procurement cost for the data set having two types of vehicles, and the full-load transportation cost is three times of the empty load transportation cost. Therefore, heuristics focusing on only procurement cost, transportation cost, or vehicle assignment cost, or even considering the total cost myopically, could result in high cost, demonstrating the importance of considering sourcing, vehicle assignment, and routing decisions simultaneously, i.e. the high value of ISDP model.

| | Data set 1 | Data set 2 | Data set 3 |
|---|---|---|---|
| Savings of ISDP model over myopic heuristic | 11.4% | 18.7% | 3.5% |

Table 2.3: Comparison between solutions to ISDP model and myopic heuristics.

Earlier discussion indicates that sequentially solving the sourcing and delivery problem can lead to an infeasible schedule, which is demonstrated in our computational test. We relax the ISDP model by excluding flow conservation constraints (2.5) and (2.6). The relaxed model provides an optimal solution to the sourcing problem with suppliers capacity constraints. From the optimal solution, we fix the suppliers customers assignment and then solve the delivery problem. For two sets of data, the sequential approach gives feasible solutions, while for the other data set, the sequential approach leads to an infeasible schedule. The infeasibility results from the suppliers delivery closing period and/or opening period with the customers due dates, requiring deliveries of more than the number of available vehicles in a single period. This result further demonstrates the importance of considering the integrated sourcing and delivery planning model.

In addition, planners benefit strategically by examining whether they have sufficient suppliers or the adequate number of vehicles to transport materials. If the company does not have enough vehicles to transport material to all the customers by the due dates, they can either purchase or rent more vehicles in advance. The suppliers know how much the customers require in each period and over the horizon. Moreover, our model can perform what-if analysis. For instance, if we have one fewer vehicle, we can know whether the due dates for customers can be met and how much the total cost is. With one more vehicle, how much can we save on the total cost? The planners can check if they can rent a vehicle with expenses less than the savings. The model provides another opportunity for cost savings. Furthermore, as contingences or new requirements arise, the ISDP model can easily provide the updated optimal schedule.

## 2.6 Conclusion

In this essay, we address the ISDP motivated by the material supply planning task for scheduled construction projects at an actual company that we worked with. The ISDP entails sourcing, vehicle assignment, and routing decisions to minimize total procurement, transportation, and vehicle assignment cost. Based on the characteristics of the problem, we formulated the ISDP as a large-scale integer programming model. The ISDP is computationally intractable, so we developed an effective solution methodology to obtain the near-optimal sourcing and delivery schedule within short time. The solution method includes preprocessing the problem to eliminate the number of variables to reduce the problem size, developing a cutting plane procedure to iteratively separating and adding violated valid inequalities to strengthen the models LP relaxation and reduce the computational efforts for branch-and-bound procedures, applying a LP-based heuristic to generate a good solution to speed up the branch and bound procedure. The application of our approach to three actual sourcing and delivery planning problems demonstrates the effectiveness of our solution methodology.

Moreover, the ISDP model provides annual cost savings of over 10% on average compared to a myopic heuristic. With the expensive construction projects, the savings are substantial. The results demonstrate the high value of the ISDP model minimizing the procurement cost, transportation cost, and vehicle assignment cost simultaneously. Furthermore, the ISDP optimization model is a valuable decision support tool for tactical planning. The model provides a comprehensive framework for decision-making by clarifying objectives and costs, and ensuring consideration of all factors such as capacities and timing constraints. From the obtained solution, we can have an overview of material production and transportation needs, sourcing patterns during the planning horizon, vehicle deployment, and delivery schedules. In addition, using the model, we are able to know beforehand if we have sufficient suppliers for the customers and the adequate number of vehicles to make deliveries on time. The model also provides useful what-if capabilities to analyze the impact or reducing the number of suppliers, vehicles, and so on. With the increasing transportation costs and trends toward more outsourcing, the ISDP model has wide applications in industry.

# Chapter 3

# Assortment Planning for Vertically Differentiated Products

## 3.1 Introduction

When choosing what products to carry in a given product category, retailers typically have to choose from hundreds, possibly thousands of variants, offered by dozens of different manufacturers. For instance, Newegg (www.newegg.com) carries more than 400 different USB flash drives and 1400 different televisions but these only represent a fraction of the existing products in the market as the retailer does not offer every brand and model. Selecting products to offer is a challenging problem for retailers, because customer choice depends on customer preferences as well as on the available assortment, and profit depends on sales and the relative profitability of products. A naive way to obtain the profit-maximizing assortment would be to enumerate all possible combinations of products and selling prices and identify the most profitable one. However, this method is not practical for popular product categories such as televisions and computers for which the number of possible products to choose from is very large. Hence, developing efficient methods to obtain a profit-maximizing assortment is very important and valuable for retailers.

This essay addresses the problem of a retailer managing a category of vertically differentiated products. Each product is characterized by a quality level. If the product has many vertical attributes, we assume that these attributes can be collapsed into a single quality dimension. All else equal, customers prefer a product with a higher quality level to a product with a lower quality level, but they differ in how they value quality. We assume that customers determine what product to buy by maximizing a linear utility function which is increasing in quality and their valuation of quality and is decreasing in price. While the retailer does not know how a specific customer values a unit of quality, she knows the distribution of customer valuations. The retailer buys the products from one or multiple manufacturers so that the quality levels of the products are exogenously determined. The retailer pays the manufacturer(s) a variable cost per product sold and incurs a fixed cost per product included in the assortment.

First, we examine the scenario where selling prices are exogenous to the retailer and her only decision is to choose the set of products to be included in the assortment. This setting is motivated by product categories for which the manufacturers' suggested retail prices (MSRP) are so prevalent that retailers do not deviate from it. Carlton & Chevalier (2001) state that while "manufacturers may not contractually bind the retailer to charge the MSRP [...] (they) may be willing to supply 'exclusive' products to retailers who adopt an across-the-board 'no discounting' policy". The data they collect about the fragrance market show that a number of stores (namely upscale beauty and department stores) are charging the MSRP for the products they sell. We show that, when prices are exogenous to the retailer, the optimal set is a function of the distribution of customer valuations for quality and can be obtained by solving a shortest path problem.

Second, we consider the case where the retailer also decides the selling price of the products offered in her assortment, i.e., selling prices are endogenous variables. We show that the optimal assortment does not depend on the distribution of customer valuations for quality, as long as this distribution has an increasing failure rate and the fixed cost is negligible. We propose a number of algorithms to obtain the optimal assortment. The complexity of these algorithms depends on the value of the fixed cost and on properties of the distribution of customer valuations for quality.

Third, we compare the optimal assortment in the two cases studied and obtain some interesting insights. In particular, we show that the optimal assortment may contain *dominated products* when prices are exogenous, but not when prices are endogenous. We say that a product is *dominated* if there exists a product with a higher quality level, lower variable cost, and (in the exogenous case only) higher selling price in the set of potential products to offer. We also demonstrate that the products included in the optimal assortment are such that the selling prices, profit margins and price-to-quality ratios are increasing in the quality level. Finally, we show that if some of the offered products become unavailable, it may be optimal for the retailer to include products which previously were not in the optimal assortment. However, it is never optimal to drop products which were offered and are still available.

An example of application for our model is that of a retailer selling USB flash drives from multiple manufacturers (such as Kingston, SanDisk, HP, etc.). These drives differ mainly in memory size (2Gb, 4Gb, 8Gb, etc.) and, at equal prices, customers prefer a drive with higher memory size to a drive with a lower memory size. Memory sizes (i.e., quality levels) are determined by the manufacturers. Each manufacturer may provide an MSRP for each variant to the retailer. Depending on her market power, the retailer may or may not be able or willing to deviate from this price. If she does not deviate, then our results from Section 3.3 apply. If the retailer determines the selling prices, our results from Section 3.4 apply.

It is important to note that the exogenous price problem is not a special case of the endogenous price problem and vice versa. First order conditions can be used to solve for prices in the endogenous prices case, but the resulting prices may be a function of the chosen assortment so that it is generally not possible to decouple the optimal prices and optimal assortment problems. As shown later, when prices are decision variables, we are able to establish more properties about the optimal assortment and develop more efficient algorithms to identify the optimal assortment.

The paper that is most closely related to ours is that of Barghava & Choudhary (2001) who also consider the problem of selecting and pricing vertically differentiated products when quality levels are exogenously determined. However, our work differs from theirs in numerous ways. First, we consider a fixed cost and show that it is an important factor in determining the optimal assortment. Second, we consider the case of exogenous selling prices. Third, we provide a full characterization of the optimal assortment when prices are endogenous. In contrast, Barghava & Choudhary (2001) only provide a partial characterization of the optimal assortment when there is no fixed cost. They focus only on two special cases: when the optimal assortment contains only the highest quality product and when it contains all the products. In other words, while their work addresses the question "when does the optimal solution have a noteworthy structure?", our work answers the more general question "how can we obtain the optimal solution for any given scenario and what are its properties?".

The rest of this essay is organized as follows. In Section 3.2 we review the related literature. In Section 3.3 we present the exogenous prices model and develop an efficient algorithm to identify the optimal assortment. Section 3.4 contains the endogenous prices model and the corresponding results. In Section 3.5 we compare the two models and discuss interesting insights. Section 3.6 concludes our work and provides directions for future research. All proofs are presented in the Appendix.

## 3.2 Literature Review

Besides Barghava & Choudhary (2001), our work is at the intersection between two streams of research: the work on vertically differentiated products and the work on assortment planning. We first review the work on vertically differentiated products, and then discuss the relevant papers on assortment planning.

Mussa & Rosen (1978) is, to our knowledge, the earliest work on vertical differentiated products. In this paper, the authors capture the heterogeneity of customers using a continuous valuation parameter $\theta$. The utility that a customer with valuation $\theta$ gets from a product of quality $q$ is $\theta q$, i.e., the utility function is linear. Assuming convex production cost, the authors show that price discrimination by offering products of different quality levels is optimal. Moorthy (1984) generalizes the utility and cost functions and uses discrete parameters to represent customer segments. He concludes that the firm may reduce the number of product versions offered in order to mitigate the cannibalization effect. Several other researchers, such as Green & Krieger (1985), and Dobson & Kalish (1988, 1993), use a limited number of customer segments to represent the market. The information about the number of customers and reservation price for products associated with each segment is assumed to be known. They formulate the problem as a mathematical programming model and develop heuristics to solve it. Assuming concave variable costs and considering a fixed setup cost for producing a batch of products, Netessine & Taylor (2007) examine how production technology impacts the optimal product line design. They demonstrate that cannibalization may lead to offering more products and higher quality in the presence of production technology.

The papers mentioned above take the point of view of a manufacturer as it is assumed that the decision maker can pick any quality level for their products and possibly provide a continuous menu of quality-price combinations. In contrast, our work takes the point of view of a retailer and assumes that there is only a discrete set of quality levels to choose from. Like these papers, our work adopts the linear utility function but we do not make any assumption about the cost function and make only limited assumptions about the distribution of customer valuations in endogenous prices case.

Several other papers which study product differentiation problems incorporate issues we do not explore here, such as the presence of outside opportunities, see Chen & Seshadri (2007), network effects on the versioning strategies; see for example Barghava & Choudhary (2004) and Jing (2007), and the optimal time to introduce product versions, i.e., either simultaneously or sequentially; see for example Moorthy & PNG (1992) and Raghunathan (2000). Finally, a number of papers consider the problem of offering vertically differentiated products in a competitive setting; see for example Shaked & Sutton (1982), Moorthy (1988), Shugan (1989), Rhee (1996), and Jing (2006).

In assortment planning problems, a firm chooses what products to offer from a discrete set of potential products and customers have heterogenous preferences for these products. The papers usually differ in the consumer choice model that is used to represent customer preferences and the type of substitution they assume. Most papers assume that products are horizontally rather than vertically differentiated, which means that two customers can have a different favorite product even if all products are offered at the same price. Since our work does not include the presence of inventory, it is most closely related to the work on assortment planning that assumes static, assortment-based substitution. So we will focus on these papers. For a broader review of the topic (including papers that assume dynamic, stock-out based substitution), see Kok *et al.* (2008).

Among the earliest papers on assortment planning, Pentico (1974) studies a one-dimensional assortment planning problem with downward substitution for stochastic demand and obtains the optimal solution with an assumption regarding the sequence of customer arrivals and a 'no crossover' assumption, which preclude dynamic substitution. Van Ryzin & Mahajan (1999) use the multinomial logit model to represent customer preferences for horizontally differentiated products and show that the optimal assortment includes a subset of the most popular products. Gaur & Honhon (2006) consider the same problem but use a locational choice model to represent customer preferences. They introduce a unimodal distribution of customers on the attribute space, and show that the products in the optimal assortment are equally spaced and need not include the most popular product. Smith & Agrawal (2000) consider this problem under stock-out based substitution but provide a solution method which assumes assortment-based substitution and use a choice model specified by first choice probabilities and a substitution matrix. Cachon *et al.* (2005) generalize the consumer choice process to incorporate search costs, and show that ignoring consumer search in demand estimation can result in an assortment with less variety and lower expected profit than the optimal solution.

The papers mentioned so far all assume that prices are given. The following papers consider prices as decision variables. Hopp & Xu (2005) use a Bayesian Logit model to study the impact of modular design on the joint assortment planning and pricing problem under assortment-based substitution. They show that the optimal assortment for a risk-averse retailer is composed of the variants with the highest price markups. Maddah & Bish (2007) consider a similar setting and propose a dominance relationship for the general case that simplifies the search for an optimal assortment. Aydin & Porteus (2008) study the joint assortment and pricing problem under price-based substitution with a demand model involving multiplicative uncertainty. Alptekinoglu & Corbett (2008) use the locational choice model to study competitive product positioning and pricing.

To the best of our knowledge, our work is the first paper that considers the problem of selecting products from a discrete set of vertically differentiated options when prices are fixed and the problem of selecting and pricing vertically differentiated products in the presence of a fixed cost, when prices are decision variables.

## 3.3 The Exogenous Prices Case

As mentioned in the Introduction, this setting applies, for example, when the retailer is not able to or not willing to deviate from the MSRP and hence, the selling prices are exogenous to the retailer. In this section, we present the model, analyze the optimal assortment structure, and develop an efficient algorithm to identify the optimal solution.

### 3.3.1 Model

We consider a product category with $n$ vertically differentiated products. Let $q_j$ denote the quality of product $j$. The quality level can also be regarded as a combination of many of the product's characteristics if the product has many characteristics, which is a common assumption in the literature on vertical differentiated products. Without loss of generality we assume that $0 \leq q_1 < q_2 < ... < q_n$.[1] Let $r_j \geq 0$ and $c_j \geq 0$ be the selling price, variable cost of product $j$ respectively. Note that we do not assume that $c_j \geq c_{j-1}$. Let $K \geq 0$ be a fixed cost incurred for each product that is offered. In practice $K$ includes, for example, the cost of advertising the product. For notational convenience, let 0 be a fictitious product 0 with $q_0 = c_0 = r_0 = 0$. Let $\vec{r} = (r_1, ..., r_n)$, $\vec{c} = (c_1, ..., c_n)$, and $\vec{q} = (q_1, ..., q_n)$.

We assume that customers are characterized by their willingness to pay for one unit of quality in the product category, or *valuation*. A customer with valuation $\theta$ gets utility $\theta q_j - r_j$ from buying one unit of product $j$ and zero for additional units. Without loss of generality, we assume that the utility of buying nothing is equal to zero. A customer buys the product which gives him the highest utility as long as it is positive. The retailer cannot identify the specific $\theta$ value for any customer, but knows the distribution of $\theta$. Let $f(\theta)$ and $F(\theta)$ with support $[\underline{\theta}, \bar{\theta}]$ denote the probability density function and cumulative density function of customer valuations respectively, where $\underline{\theta} \geq 0$ and $0 < \bar{\theta} \leq \infty$.

In the exogenous prices model, $\vec{r}, \vec{q}$ and $\vec{c}$ are given and fixed. It is assumed that $c_j < r_j < q_j$ for $j = 1, ..., n$; otherwise, it would be optimal not to stock product $j$ for sure. Note that we do not exclude the possibility that two products

[1] All of our results would continue to hold if we had $q_j = q_{j+1}$ for some $j$ but, for ease of exposition, we ignore this case.

$i, j$ are such that $r_i < r_j$ and $q_i > q_j$ (better quality for a lower price). The retailer's decision is to determine which products to offer. Let $S$ denote the set of products that are offered, or *assortment*. We summarize our notation in Table 3.1.

| Symbol | Definition |
|---|---|
| $n$ | Number of potential products in the category |
| $j$ | Product index, $j = 1, ..., n$ |
| $q_j$ | Quality level of product $j$ |
| $c_j$ | Variable cost of product $j$ |
| $r_j$ | Selling price of product $j$ |
| $\vec{q}$ | Quality vector |
| $\vec{c}$ | Variable cost vector |
| $\vec{r}$ | Selling price vector |
| $\theta$ | Consumer valuation, $\in [\underline{\theta}, \bar{\theta}]$ |
| $f(\theta)$ | Probability density function of consumer valuation |
| $F(\theta)$ | Cumulative distribution function of consumer valuation |
| $h(\theta)$ | Failure rate of distribution $F(\theta)$ |
| $\eta(\theta)$ | Inverse failure rate |
| $\theta_j$ | Valuation of consumer who is indifferent between products $j-1$ and $j$ |
| $P_j$ | Purchase probability of product $j$ |
| $S$ | Assortment, i.e., set of products offered |
| $S^*$ | Optimal assortment |
| $K$ | Fixed cost incurred for each product included in the assortment |
| $\mathbb{E}\Pi$ | Retailer's expected profit |
| $\mathbb{E}\Pi^*$ | Retailer's optimal expected profit |

Table 3.1: Notation

Let $P_j(S)$ be the proportion of customers who purchase product $j$ given assortment $S$, or *purchase probability*. We have:

$$P_j(S) = \begin{cases} \int_{\underline{\theta}}^{\bar{\theta}} I\{q_j\theta - r_j = \max_{i\in S}(q_i\theta - r_i) \text{ and } q_j\theta - r_j > 0\} dF(\theta) & \text{if } j \in S \\ 0 & \text{otherwise} \end{cases}$$

where $I\{A\}$ is the indicator function for event $A$.

In theory, it is possible to have $P_j(S) = 0$ for $j \in S$, that is, product $j$ is offered but no customer buys it. In this case, removing product $j$ does not affect the demand for other products and it decreases the total fixed cost; therefore, the optimal assortment never includes a product with zero purchase probability. It follows that one can restrict the search for the optimal assortment to sets $S = \{j_1, j_2, ..., j_m\}$ such that $j_1 < j_2 < ... < j_m$ and

$$\underline{\theta} \leq \frac{r_{j_1}}{q_{j_1}} < \frac{r_{j_2} - r_{j_1}}{q_{j_2} - q_{j_1}} < ... < \frac{r_{j_m} - r_{j_{m-1}}}{q_{j_m} - q_{j_{m-1}}} < \bar{\theta}, \tag{3.1}$$

which are necessary and sufficient conditions for $P_{j_i}(S) > 0$ for $i = 1, ..., m$ since

$$P_{j_i}(S) = F\left(\frac{r_{j_{i+1}} - r_{j_i}}{q_{j_{i+1}} - q_{j_i}}\right) - F\left(\frac{r_{j_i} - r_{j_{i-1}}}{q_{j_i} - q_{j_{i-1}}}\right) \text{ for } i = 1, ..., m-1, \tag{3.2}$$

$$P_{j_m}(S) = 1 - F\left(\frac{r_{j_m} - r_{j_{m-1}}}{q_{j_m} - q_{j_{m-1}}}\right). \tag{3.3}$$

where $j_0 = 0$. Figure 3.1 illustrates an example with three products. Consumers in Group $A$ are such that $\underline{\theta} \leq \theta \leq \frac{r_1}{q_1}$ and purchase nothing since they get a non-positive utility from every product. Consumers in Group $B$ are such that $\frac{r_1}{q_1} < \theta \leq \frac{r_2-r_1}{q_2-q_1}$ and purchase product 1 because it gives them the highest utility. Consumers in Group $C$ are such that $\frac{r_2-r_1}{q_2-q_1} < \theta \leq \frac{r_3-r_2}{q_3-q_2}$ and buy product 2 although each product gives them a positive utility. Finally, consumers in group $D$ are such that $\frac{r_3-r_2}{q_3-q_2} < \theta \leq \overline{\theta}$ and purchase product 3. Hence, the purchase probabilities are equal to $F\left(\frac{r_2-r_1}{q_2-q_1}\right) - F\left(\frac{r_1}{q_1}\right)$, $F\left(\frac{r_3-r_2}{q_3-q_2}\right) - F\left(\frac{r_2-r_1}{q_2-q_1}\right)$ and $1 - F\left(\frac{r_3-r_2}{q_3-q_2}\right)$ respectively for product 1, 2 and 3.

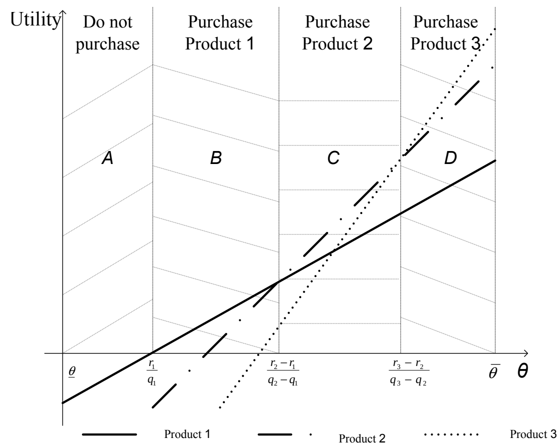


FIGURE 3.1: Purchase probabilities in an example with 3 products.

Without loss of generality we normalize mean demand to 1.[2] The expected profit is as follows:

$$\mathbb{E}\Pi(S) \;=\; \sum_{j\in S} P_j(S)(r_j - c_j) - K|S|, \tag{3.4}$$

where $|S|$ denotes the size of set $S$. The firm's objective is to find $S^*$ such that

$$\mathbb{E}\Pi^* = \mathbb{E}\Pi(S^*) \;=\; \max_{S\subseteq\{1,\ldots,n\}} \mathbb{E}\Pi(S). \tag{3.5}$$

### 3.3.2 Results

To solve the retailer's profit maximization problem (3.5), we can enumerate all possible assortments $S$, compute their expected profit, and identify the optimal assortment with the highest expected profit. However, the complexity of

[2]If mean demand is $\mu > 1$, then $K$ is set to equal to the fixed cost of stocking a product divided by $\mu$.

this naive enumeration method is $O(2^n)$, which is not practical when $n$ is large.

Consequently, we develop an efficient algorithm to obtain the optimal assortment. This algorithm is based on some important properties of the optimal assortment which we present first. Let $S^* = \{j_1, j_2, ..., j_m\}$ be the optimal set such that $j_1 < j_2 < ... < j_m$.

*Lemma* 3.3.1. The products in an optimal assortment $S^*$ satisfy the following three conditions:

$$r_{j_1} < r_{j_2} < ... < r_{j_m}, \tag{3.6}$$

$$\underline{\theta} \leq \frac{r_{j_1}}{q_{j_1}} < \frac{r_{j_2}}{q_{j_2}} < ... < \frac{r_{j_m}}{q_{j_m}} < \bar{\theta}, \tag{3.7}$$

$$r_{j_1} - c_{j_1} \leq r_{j_2} - c_{j_2} \leq ... \leq r_{j_m} - c_{j_m}. \tag{3.8}$$

Lemma 3.3.1 indicates that, in the optimal assortment, the product prices and price-quality ratios are strictly increasing in the quality level and the profit margins are non-decreasing in the quality levels. The first two conditions are necessary for all products in $S^*$ to get a positive purchase probability. To prove the third property we show that, if a product $j$ has a lower profit margin than products of lower quality, then the retailer can get a higher expected profit by removing product $j$.

From (3.2), (3.3) and (3.4), notice that the contribution of product $j_i$ to the expected profit in assortment $S$ depends only on the adjacent products $j_{i-1}$ and $j_{i+1}$, since products before $j_{i-1}$ and after $j_{i+1}$ have no impact on the purchase probability of product $j_i$. As a consequence, we are able to model this problem as a shortest path problem and solve it in polynomial time. Moreover, we use the properties of Lemma 3.3.1 to construct a parsimonious network.

We construct a graph $G = (V, A)$, where $V$ is the set of nodes and $A$ is the arc set of $G$. Node set $V$ consists of pairs of products $(i, j)$ such that product $j$ has a higher quality and price-quality ratio than product $i$ and a profit margin which is is no lower than that of product $i$. A node $(i, j) \in V$ indicates that products $i$ and $j$ could be offered together in the assortment. We also introduce two fictitious nodes: a source node $(0, 0)$ and a destination node $(n + 1, n + 1)$. If nodes $(i, j)$ and $(j, k)$ satisfy $\underline{\theta} \leq \frac{r_j - r_i}{q_j - q_i} < \frac{r_k - r_j}{q_k - q_j} < \bar{\theta}$, then the arc between these two nodes is a valid arc, which belongs to $A$. A valid arc between $(i, j)$ and $(j, k)$ implies that product $j$ could be offered along with products $i$ and $k$. We are able to compute the cost of the arc between $(i, j)$ and $(j, k)$ using the prices, variable costs, and quality levels of these three products. We find the optimal assortment by solving the shortest path problem from the source node to the destination node. We formalize the procedure as follows.

**ALGORITHM: Shortest Path 'Exo'**

- Step 1. Construct the node set $V$, which consists of the following nodes:

$$
\begin{aligned}
V \;=\; & \left\{(0, i) : 1 \leq i \leq n \text{ and } \underline{\theta} \leq \frac{r_i}{q_i} < \bar{\theta}\right\} \cup \{(j, n+1) : 1 \leq j \leq n\} \\
& \cup \{(0, 0), (n+1, n+1)\} \\
& \cup \left\{(i, j) : 1 \leq i < j \leq n \text{ and } \frac{r_i}{q_i} < \frac{r_j}{q_j} < \bar{\theta} \text{ and } r_i - c_i \leq r_j - c_j\right\}
\end{aligned}
$$

- Step 2. Construct the arc set by adding an arc from node $(i, j) \in V$ to $(l, k) \in V$ to set $A$ if $j = l = k$ or $j = l < k$ and

$$\begin{cases} \underline{\theta} \leq \frac{r_j - r_i}{q_j - q_i} < \frac{r_k - r_l}{q_k - q_l} < \bar{\theta} & \text{if } k < n+1 \\ \underline{\theta} \leq \frac{r_j - r_i}{q_j - q_i} < \bar{\theta} & \text{if } k = n+1 \end{cases}$$

- Step 3. Compute the arc costs:

$$C_{(i,j),(j,k)} = \begin{cases} K - (r_j - c_j)\left[F\left(\frac{r_k - r_j}{q_k - q_j}\right) - F\left(\frac{r_j - r_i}{q_j - q_i}\right)\right] & \text{if } 0 < j < k < n+1 \\ K - (r_j - c_j)\left[1 - F\left(\frac{r_j - r_i}{q_j - q_i}\right)\right] & \text{if } 0 < j < k = n+1 \\ 0 & \text{otherwise} \end{cases}$$

- Step 4. Solve the shortest path problem from $(0, 0)$ to $(n + 1, n + 1)$.

*Theorem* 3.3.1. The Shortest Path 'Exo' algorithm gives an optimal assortment.

Note that it is necessary to define the nodes as pairs of products as one needs to know which product is to the left and which product is to the right of a given product in order to compute its contribution to the expected profit. Also, note that our result does not make any assumption on the distribution of customer valuation $F$. In the context of a firm determining the optimal tradeoff between variety and leadtime for horizontally differentiated products, Alptekinoglu & Corbett (2009) also use a shortest path formulation to solve for the optimal product line.

*Corollary* 3.3.2. The complexity of the Shortest Path 'Exo' algorithm is $O(n^3)$.

The efficiency of the Shortest Path 'Exo' algorithm makes our method to identify the optimal assortment attractive. We use an example in next section to illustrate how to use the algorithm.

### 3.3.3 Example and Properties of the Optimal Solution

The shortest path 'Exo' algorithm is most useful when the retailer faces a large number of candidate products, but the small example below illustrates the underlying mechanism and provides valuable insights.

*Example* 3.1. A retailer can choose from three vertically differentiated products with $\vec{c} = (5, 4.5, 50)$, $\vec{q} = (30, 36, 100)$ and $\vec{r} = (15, 15.5, 80)$. She knows that the distribution of customer valuations follows distribution $F(\theta) = 1-(1-\theta)^b$, where $b > 0$ with support $[0, 1]$. She needs to decide what products to offer in order to maximize the profit.

The distribution of customer valuations $F(\theta) = 1 - (1 - \theta)^b$ is a common distribution to model consumer preferences; see, for example, Debo *et al.* (2005) and Sundararajan (2004). It corresponds to a beta distribution[3] with parameter $a = 1$ and $b > 0$. Note that the uniform distribution on $[0, 1]$ is a special case of this distribution obtained by setting $b = 1$. If $b > 1$, then the distribution function is concave, meaning that there are more customers with low valuations. If $0 < b < 1$, then the distribution function is convex, meaning that there are more customers with high valuations.

Using the enumeration method, one would need to consider seven assort-

[3]The probability density function is $B(\theta; a, b) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{\beta(a,b)}$, where $\beta(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1}dx$.

ments: $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}$, and $\{1,2,3\}$ and compute the expected profit for each one. Figure 3.2 shows the graph. (0,0) and (4,4) are the source and destination nodes respectively.

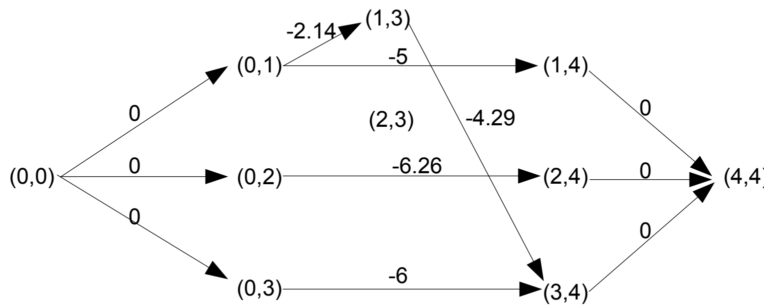


FIGURE 3.2: Graph for Example 3.1 with $K = 0$ and $b = 1$.

Table 3.2 shows the correspondence between the valid path and assortment. The network contains four valid paths which correspond to the following assortments: $\{1\}, \{2\}, \{3\}$, and $\{1,3\}$. Therefore, compared to the enumeration method, we are able to exclude three assortments.

| Valid path | Assortment |
|---|---|
| $(0,0) \to (0,1) \to (1,3) \to (3,4) \to (4,4)$ | $\{1,3\}$ |
| $(0,0) \to (0,1) \to (1,4) \to (4,4)$ | $\{1\}$ |
| $(0,0) \to (0,2) \to (2,4) \to (4,4)$ | $\{2\}$ |
| $(0,0) \to (0,3) \to (3,4) \to (4,4)$ | $\{3\}$ |

TABLE 3.2: Mapping between path and assortment for Example 3.1.

In Figure 3.2, the arc costs are computed using $K = 0$ and $b = 1$. In this case, applying the *Shortest Path 'Exo' algorithm* gives the optimal path $(0,0) \to (0,1) \to (1,3) \to (3,4) \to (4,4)$, which corresponds to the assortment $\{1,3\}$, and gives an optimal expected profit of \$6.43.

We now compare the optimal assortments for different values of the fixed cost, i.e., $K = 0$ or $0.5$, and different distributions of customer valuations, i.e., $b = 1/2, 1$ or $2$. Table 3.3 shows the optimal assortment $S^*$ for each possible combination of $K$ and $b$. We see that the optimal assortment varies with the customer valuation distribution $F$ since it varies with $b$ (we will show that this result is not true when $K = 0$ in the endogenous prices case). In particular, when $b = 1/2$, many customers have a high valuation of quality and it is optimal to offer only product 3, which is the most profitable product. When $b = 1$, the distribution of quality valuation is uniform and it is optimal to add product 1 to the assortment in order to increase total demand, even though it also leads to some customers switching from product 3 to the less profitable product 1. Finally when $b = 2$ and most customers have low valuations of quality and it is most important to capture the greatest possible market, so the optimal assortment is to offer only product 2, which has the lowest price-to-quality ratio.

When $b = 1$, we see that, as the fixed cost changes from $K = 0$ to $K = 0.5$, the optimal assortment changes from $\{1, 3\}$ to $\{2\}$. Therefore, retailers need to be aware of any change in the fixed cost as it can have a great impact on the optimal assortment.

As expected, the expected profit decreases with $b$ since a decrease in $b$ implies that more customers have high valuation for quality.

Another interesting observation is with regard to the relative attractiveness of the products the optimal assortment. We say that product $i$ *dominates* product $j$ if $c_i < c_j$, $q_i > q_j$, $r_i > r_j$, and $\frac{r_i}{q_i} < \frac{r_j}{q_j}$ and that a product is *dominated*

| | $b = 1/2$ | $b = 1$ | $b = 2$ |
|---|---|---|---|
| $K = 0$ | $S^* = \{3\}$ | $S^* = \{1, 3\}$ | $S^* = \{2\}$ |
| | $\mathbb{E}\Pi^* = 13.41$ | $\mathbb{E}\Pi^* = 6.43$ | $\mathbb{E}\Pi^* = 3.57$ |
| $K = 0.5$ | $S^* = \{3\}$ | $S^* = \{2\}$ | $S^* = \{2\}$ |
| | $\mathbb{E}\Pi^* = 12.91$ | $\mathbb{E}\Pi^* = 5.74$ | $\mathbb{E}\Pi^* = 3.07$ |

TABLE 3.3: Optimal assortments and optimal expected profit values as a function of $K$ and $b$ in Example 3.1.

if there exists at least one product in $\{1, ..., n\}$ that dominates it. In Example 3.1, product 1 is dominated by product 2 and yet product 1 is included in the optimal solution when $b = 1$ and $K = 0$. It follows that one cannot eliminate dominated products as they might be included in the optimal assortment (in the next section we show that this property does not hold in the endogenous prices case).

## 3.4 The Endogenous Prices Case

This section presents the model when prices are endogenous. We discuss the optimal solution structure and develop several efficient algorithms to identify the optimal assortment. The choice of which algorithm to use depends on the value of the fixed cost and the nature of the distribution of customer valuations.

### 3.4.1 Model

In the endogenous prices model, only $\vec{q}$ and $\vec{c}$ are given and fixed. The retailer needs to determine the assortment $S$ and the selling price $r_j$ for product $j \in S$. Note that it is possible to set the selling price of a product so high than no customers buys it, that is, such that the product has a zero purchase probability. In this case, the product should not be included in the assortment due to the fixed cost. Therefore we regard $\vec{r}$ as the only decision variable in this problem and define the corresponding assortment as the set of the products with positive purchasing probability given $\vec{r}$. As in the exogenous model, each customer observes $\vec{r}$ and $\vec{q}$ then chooses the product that gives him the highest utility as long as it is positive.

Let $h(\theta) = \frac{f(\theta)}{1-F(\theta)}$ be the *failure (or hazard) rate* of distribution $F$, and $\eta(\theta) = \frac{1}{h(\theta)}$ be the *inverse failure (or hazard) rate*. $F$ is an *increasing failure rate* (IFR) distribution if $h'(\theta) \geq 0$ or equivalently $\eta'(\theta) \leq 0$ for all $\theta$. In this section, we assume that $F$ is an IFR distribution. This assumption is satisfied by most common distributions, e.g., uniform, normal, logistic, chi-squared, exponential, Laplace, and beta distributions with $a = 1$. Note that this assumption is not required in the exogenous prices case.

Let $\theta_j = \frac{r_j - r_{j-1}}{q_j - q_{j-1}}$ for $j = 1, ..., n$. A customer with valuation $\theta_j$ gets the same utility from products $j-1$ and $j$. Without loss of generality, we assume that the prices are set such that

$$\underline{\theta} \leq \theta_1 \leq ... \leq \theta_n \leq \bar{\theta}. \tag{3.9}$$

because for any set of prices that does not satisfy this condition, there exists a set of prices that does, with the same purchase probability for each product and the same total expected profit.

Let $P_j(\vec{r})$ be the purchase probability for product $j$. We have:

$$
\begin{aligned}
P_j(\vec{r}) &= F(\theta_{j+1}) - F(\theta_j), \\
P_n(\vec{r}) &= 1 - F(\theta_n).
\end{aligned}
$$

Let $S(\vec{r})$ denote the assortment, that is, the set of products with $P_j(\vec{r}) > 0$ given price vector $\vec{r}$. Given $(\theta_1, ..., \theta_n)$ satisfying (4.1), we have $S(\vec{r}) = \{j = 1, ..., n : \underline{\theta} \leq \theta_j < \theta_{j+1} < \bar{\theta}\}$. We write the expected profit as

$$
\mathbb{E}\Pi(\vec{r}) = \sum_{j=1}^{n} P_j(\vec{r})(r_j - c_j) - K|S(\vec{r})|.
$$

The firm's objective is to find $\vec{r}^*$ such that

$$
\mathbb{E}\Pi^* = \mathbb{E}\Pi(\vec{r}^*) = \max_{\vec{r}} \mathbb{E}\Pi(\vec{r}).
$$

We use $S^* = S(\vec{r}^*)$ to denote the optimal assortment.

Note that there is a one-to-one correspondence between $\vec{r}$ and $\vec{\theta} = (\theta_1, ..., \theta_n)$. Let $\vec{\theta}$ be the vector corresponding to $\vec{r}$, and $S(\vec{\theta})$ be the assortment corresponding to $\vec{\theta}$. Hence, we can rewrite the expected profit function as a function of $\vec{\theta}$ only as:

$$
\mathbb{E}\Pi(\vec{\theta}) = \sum_{j=1}^{n} [1 - F(\theta_j)] \left[\theta_j(q_j - q_{j-1}) - (c_j - c_{j-1})\right] - K|S(\vec{\theta})|.
$$

The retailer's profit maximization problem is

$$\begin{array}{l}\max_{\vec{\theta}} \mathbb{E}\Pi(\vec{\theta}) \\ s.t. \;\; \underline{\theta} \leq \theta_1 \leq \theta_2 \leq ... \leq \theta_n \leq \bar{\theta}\end{array} \tag{3.10}$$

### 3.4.2 Results

In Section 3.3.2, we observe that, when prices are exogenous, the products in the optimal assortment have some nice properties regarding prices and price-quality ratios. Those properties enable us to develop an efficient algorithm to identify the optimal set of products to offer. Similarly, when prices are decision variables, we obtain properties of the optimal assortment then use them to develop efficient solution methods. Let $S^* = \{j_1, ..., j_m\}$, such that $j_1 < j_2 < ... < j_m$ be an optimal assortment of products with positive purchase probability.

*Lemma* 3.4.1. The products in an optimal assortment $S^*$ satisfy the following three conditions:

$$\frac{c_{j_1}}{q_{j_1}} < \frac{c_{j_2} - c_{j_1}}{q_{j_2} - q_{j_1}} < ... < \frac{c_{j_m} - c_{j_{m-1}}}{q_{j_m} - q_{j_{m-1}}} < \bar{\theta}, \tag{3.11}$$

$$\frac{c_{j_1}}{q_{j_1}} < \frac{c_{j_2}}{q_{j_2}} < ... < \frac{c_{j_m}}{q_{j_m}} < \bar{\theta}, \tag{3.12}$$

$$c_{j_1} < c_{j_2} < ... < c_{j_m}. \tag{3.13}$$

Lemma 3.4.1 shows that both the variable costs and cost-quality ratios of products in the optimal assortment are strictly increasing in the quality levels. These conditions allow the retailer to price the products so that she can extract maximum surplus from consumers.

Lemma 3.4.2 provides a method to compute the optimal prices $r^*$, once $S^*$ is known.

*Lemma* 3.4.2. The values of $\theta^*_{j_i}$ for $i = 1, ..., m$ are obtained by solving:

$$\theta^*_{j_i} \;=\; \eta(\theta^*_{j_i}) + \frac{c_{j_i} - c_{j_{i-1}}}{q_{j_i} - q_{j_{i-1}}} \qquad \text{for } i = 1, ..., m. \tag{3.14}$$

where $j_0 = 0$. For $k \notin S^*$, we have $\theta^*_k = \theta^*_{k+1}$ if $k < j_m$ and $\theta^*_k = \bar{\theta}$ if $k > j_m$. The optimal prices are obtained as follows:

$$r^*_{j_i} \;=\; \sum_{x=1}^{i} \theta^*_{j_x}(q_{j_x} - q_{j_{x-1}}), \text{ for } i = 1, ..., m. \tag{3.15}$$

Note that products which are not in $S^*$ are not offered and therefore Lemma 3.4.2 does not provide a selling price for them. Using these optimal prices, we establish some properties of the optimal assortment.

*Corollary* 3.4.3. The products in an optimal assortment $S^*$ satisfy the following two conditions:

$$r^*_{j_1} < r^*_{j_2} < ... < r^*_{j_m}, \tag{3.16}$$

$$r^*_{j_1} - c_{j_1} < r^*_{j_2} - c_{j_2} < ... < r^*_{j_m} - c_{j_m}. \tag{3.17}$$

Corollary 3.4.3 reveals that, in an optimal assortment, both product prices and profit margins are strictly increasing in the quality levels. Intuitively, if the profit margin of a product is less than another product with lower quality, the retailer can improve the profit by either changing the product price or removing the product.

In the exogenous prices case, we notice that the optimal assortment may include dominated products and therefore these products cannot be deleted before using the Shortest Path 'Exo' algorithm. We now examine if the same result holds when prices are endogenous. Because prices are no longer fixed, we have to modify the definition of dominated products slightly and say that product $i$ *dominates* product $j$ if $c_i < c_j$, $q_i > q_j$, and a product is *dominated* if there exists at least one product that dominates it. The following result shows that dominated products are never included in the optimal assortment.

*Lemma* 3.4.4. An optimal assortment $S^*$ does not contain any dominated product.

This result is particularly useful if the number of products $n$ is very large, since one can eliminate dominated products before using the algorithm below. Identifying dominated products can be done in $O(n^2)$.

In a different context, Deneckere & McAfee (1996) show that, under some conditions, it is profitable to introduce a product with a lower quality and higher variable cost than an existing product. However, they note in Lemma 3 on page 168, that this result does not hold when the ratio of how a customer values the product with low quality to how he values the product with high quality is nondecreasing in $\theta$. In our model, that ratio is equal to $\frac{\theta q_l}{\theta q_h} = \frac{q_l}{q_h}$ if $q_l$ and $q_h$ are the value of low quality and high quality respectively, which is nondecreasing in

$\theta$. Therefore, their result does not apply to our setting.

Using the properties from Lemma 3.4.1, we develop a method to find the optimal solution in polynomial time by modeling the problem of finding $S^*$ as a shortest path problem.

**ALGORITHM: Shortest Path 'Endo'**

- Step 1. Construct node set $V$, which consists of the following nodes:

$$\begin{aligned} V &= \left\{(0,i) : 1 \leq i \leq n \text{ and } \frac{c_i}{q_i} < \bar{\theta}\right\} \cup \{(j,n+1) : 1 \leq j \leq n \text{ and } \frac{c_j}{q_j} < \bar{\theta}\} \\ &\cup \{(0,0),(n+1,n+1)\} \cup \left\{(i,j) : 1 \leq i < j \leq n \text{ and } \frac{c_i}{q_i} < \frac{c_j}{q_j} < \bar{\theta}\right\}, \end{aligned}$$

- Step 2. Construct arc set by adding an arc from node $(i,j) \in V$ to $(l,k) \in V$ to set $A$ if $j = l = k$ or $j = l < k$ and

$$\begin{cases} \frac{c_j - c_i}{q_j - q_i} < \frac{c_k - r_l}{c_k - q_l} < \bar{\theta} & \text{if } k < n+1 \\ \frac{c_j - c_i}{q_j - q_i} < \bar{\theta} & \text{if } k = n+1 \text{ and } l < k \end{cases}$$

- Step 3. Compute the arc costs:

$$C_{(i,j),(j,k)} = \begin{cases} K - \left[\theta_k(q_k - q_j) - (c_k - c_j)\right]\left[1 - F(\theta_k)\right] & \text{if } k < n+1 \\ 0 & \text{if } k = n+1 \end{cases}$$

  where $\theta_k$ is the solution to $\theta_k = \eta(\theta_k) + \frac{c_k - c_j}{q_k - q_j}$.

- Step 4. Solve the shortest path problem from $(0,0)$ to $(n+1,n+1)$.

*Theorem* 3.4.1. Shortest Path 'Endo' Algorithm gives an optimal assortment.

*Corollary* 3.4.5. The complexity of the Shortest Path 'Endo' algorithm is $O(n^3)$.

Once $S^*$ is obtained by solving *Shortest Path 'Endo' Algorithm*, we can use Lemma 3.4.2 to obtain the optimal prices. Note that the *Shortest Path 'Endo' Algorithm* has the same complexity as the *Shortest Path 'Exo' Algorithm* of Section 3.3. However, in practice, the number of nodes and arcs is generally greater in the endogenous case since there are fewer conditions on the node set, and, as a result, solving the *Shortest Path 'Endo' Algorithm* generally takes longer than solving the *Shortest Path 'Exo' Algorithm*.

### 3.4.3 Example, Properties of the Optimal Solution and Special Cases

To illustrate how the *Shortest Path 'Endo' Algorithm* works, we show how to obtain the optimal assortment using the same data as in Example 3.1.

*Example* 3.2. (Cont'd from Example 3.1) Let $\vec{c}$, $\vec{q}$ and the customer valuation distribution $F$ be as in Example 3.1. The retailer needs to decide what products to offer and how to price the offered products.

In this case, the network, shown in Figure 3.3, contains ten nodes and five valid paths. Table 3.4 maps the valid paths to assortments $\{1\}, \{2\}, \{3\}$, $\{1, 3\}$,and $\{2, 3\}$, so that there is one more valid path than in the exogenous case. Compared to the enumeration method, we are able to exclude two assortments.

In Figure 3.3, the arc costs are computed using $K = 0$ and $b = 1$. In this case, applying the *Shortest Path 'Endo' algorithm* gives the optimal path

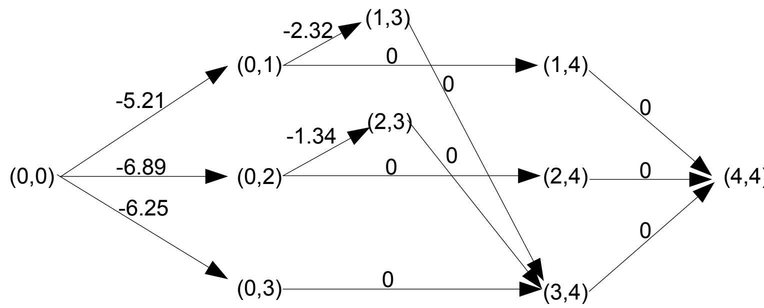


FIGURE 3.3: Graph for Example 3.2 with $K = 0$ and $b = 1$.

| Valid path | Assortment |
|---|---|
| $(0,0) \to (0,1) \to (1,3) \to (3,4) \to (4,4)$ | $\{1,3\}$ |
| $(0,0) \to (0,2) \to (2,3) \to (3,4) \to (4,4)$ | $\{2,3\}$ |
| $(0,0) \to (0,1) \to (1,4) \to (4,4)$ | $\{1\}$ |
| $(0,0) \to (0,2) \to (2,4) \to (4,4)$ | $\{2\}$ |
| $(0,0) \to (0,3) \to (3,4) \to (4,4)$ | $\{3\}$ |

TABLE 3.4: Mapping between paths and assortments in Example 3.2.

$(0,0) \to (0,2) \to (2,3) \to (3,4) \to (4,4)$ which corresponds to the assortment $\{2,3\}$ and gives an optimal expected profit of \$8.23.

As in exogenous prices case, we now compare the optimal assortments for different values of the fixed cost, i.e., $K = 0$ or $0.5$ and different distributions of customer valuations, i.e., $b = 1/2, 1$ or $2$. Table 3.5 shows the optimal assortment $S^*$ for each possible combination of $K$ and $b$.

As expected, we see that expected profit is decreasing in $K$ and $b$. Notice that the optimal assortment varies with $b$ when $K > 0$, but not when $K = 0$. Lemma 3.4.6 shows that this result for $K = 0$ is true in general.

*Lemma* 3.4.6. If $K = 0$, $S^*$ is optimal if and only if the conditions from Lemma

| | $b=1/2$ | $b=1$ | $b=2$ |
|---|---|---|---|
| $K=0$ | $S^*=\{2,3\}$<br>$r_2^*=20.25$, $r_3^*=75$<br>$\mathbb{E}\Pi^*=13.93$ | $S^*=\{2,3\}$<br>$r_2^*=20.25$, $r_3^*=75$<br>$\mathbb{E}\Pi^*=8.23$ | $S^*=\{2,3\}$<br>$r_2^*=20.25$, $r_3^*=75$<br>$\mathbb{E}\Pi^*=3.21$ |
| $K=0.5$ | $S^*=\{2,3\}$<br>$r_2^*=20.25$, $r_3^*=75$<br>$\mathbb{E}\Pi^*=12.93$ | $S^*=\{2,3\}$<br>$r_2^*=20.25$, $r_3^*=75$<br>$\mathbb{E}\Pi^*=7.23$ | $S^*=\{2\}$<br>$r_2^*=20.25$<br>$\mathbb{E}\Pi^*=2.51$ |

TABLE 3.5: Optimal assortments, price vectors, and expected profit values as a function of $K$ and $b$ in Example 3.2.

3.4.1 are satisfied along with,

$$\frac{c_k - c_{j_i}}{q_k - q_{j_i}} \geq \frac{c_{j_{i+1}} - c_k}{q_{j_{i+1}} - q_k}, \qquad \text{for } k = j_i + 1, ..., j_{i+1} - 1, \tag{3.18}$$

$$\frac{c_k - c_{j_m}}{q_k - q_{j_m}} \geq \bar{\theta}, \qquad \text{for } k = j_m + 1, ..., n. \tag{3.19}$$

Further, the optimal assortment $S^*$ is unique.

Lemma 3.4.6 implies that the products which are not included in $S^*$ have a higher cost-quality ratio than products in $S^*$ with a lower quality level. Moreover, only one assortment satisfies these conditions (remember that $S^*$ is defined as the optimal set of products with strictly positive purchase probability). It follows that the optimal assortment remains the same whatever the distribution of customers' valuation is (as long as it is IFR) as stated in Corollary 3.4.7.

*Corollary* 3.4.7. If $K = 0$, the optimal set $S^*$ does not depend on the distribution of customer valuation $F$.

This is a surprising result since, in the *Shortest path 'Endo'* algorithm, the non-zero arc costs depend on $F$, therefore, one would expect the shortest path and the corresponding assortment to vary with $F$ as in the exogenous prices case. We provide an intuition for this result in our next section.

For the case when $K = 0$, we use the properties in Lemma 3.4.6 to develop a more efficient algorithm. For each product $i$, starting with a fictitious product 0, we identify the next product to be included in the optimal assortment by looking for a product which satisfies conditions (3.11) and (4.4). We examine candidate products one by one, starting from the highest quality product (i.e., product $n$) down to product $i+1$. After identifying the optimal set $S^*$, we compute the optimal prices using Lemma 3.4.2. The algorithm is formally stated as follows.

**ALGORITHM: Zero Fixed Cost Algorithm**

- Step 0. $S^* = \emptyset$, $i = 0$.
- Step 1. If $i < n$,

  For $j := n$ down to $i+1$,

  If $\frac{c_j - c_i}{q_j - q_i} < \bar{\theta}$ AND $\frac{c_k - c_i}{q_k - q_i} > \frac{c_j - c_k}{q_j - q_k}$ for $k = i+1, ..., j-1$

  $S^* := S^* \cup \{j\}$, $i := j$ and back to step 1.

  end

  end

  end
- Step 2: Use Lemma 3.4.2 to obtain $\vec{r}^*$.

*Proposition* 3.4.8. The set $S^*$ obtained from *Zero Fixed Cost Algorithm* is optimal when $K = 0$.

*Corollary* 3.4.9. The complexity of the *Zero Fixed Cost Algorithm* is $O(n^2)$.

In practice the *Zero Fixed Cost algorithm* can be used whenever the advertising costs and other fixed cost are negligible, which is more likely to be true for online retailers. Barghava & Choudhary (2001) previously studied this setting. They provide conditions under which the optimal assortment contains all $n$ products, i.e., $S^* = \{1, ..., n\}$ and conditions under which it contains only the product with the highest quality, i.e., $S^* = \{n\}$. In contrast, our work provides an efficient algorithm to identify the optimal solution for any $(\vec{c}, \vec{q})$.

There is a nice graphical interpretation for the optimal solution when $K = 0$: on a two dimensional graph the functions $\theta q_j - c_j$ for $j = 1, ..., n$ are drawn as a function of $\theta$, the optimal assortment corresponds to the set of products that belong to the upper envelope of the lines in the positive quadrant for $\theta \in [\underline{\theta}, \overline{\theta}]$. Figure 3.4 shows the corresponding graph for Example 3.2 with $K = 0$ and $b = 1$. From the graph, we can see the optimal assortment is $\{2, 3\}$ since these two products belong to the upper envelope. We formalize this observation in the following Corollary.

*Corollary* 3.4.10. When $K = 0$, $j \in S^*$ if and only if there exists at least two values of $\theta \in [\underline{\theta}, \overline{\theta}]$ such that $\theta q_j - c_j = \max_{i=1,...,n}(\theta q_i - c_i)$.

In Example 3.2, we observe that, if for a given value of $b$, a product is offered when $K = 0$ and $K > 0$, then its optimal price is the same. Next we

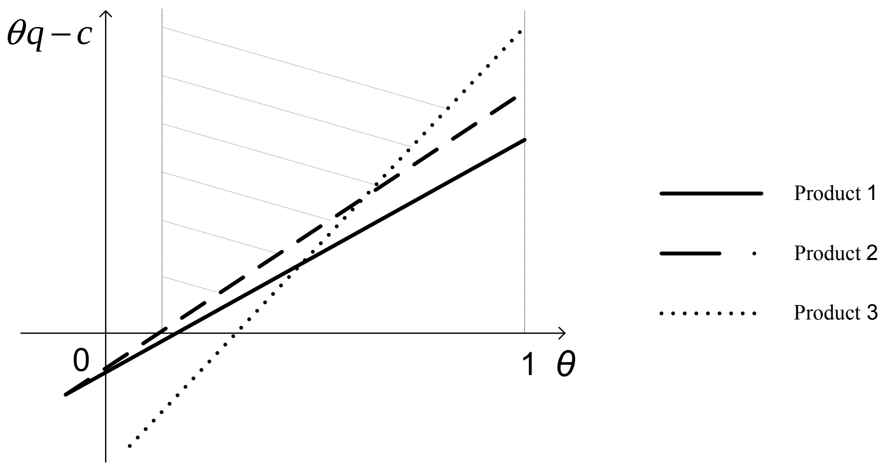


FIGURE 3.4: Graphical interpretation for $S^*$ in Example 3.2 with $K = 0$ and $b = 1$.

formally prove that this property holds whenever the inverse hazard rate function $\eta(\theta)$ is linear in $\theta$. Examples of distributions having linear inverse hazard rate include exponential distributions and $F = 1 - (1-\theta)^b$ with $b > 0$.

*Lemma* 3.4.11. Suppose $\eta(\theta) = \alpha\theta + \beta$, where $\alpha \leq 0$ and $\beta$ are constants. The optimal price of any product in $S^* = \{j_1, ..., j_m\}$ with $j_1 < ... < j_m$ depends only on its own cost and quality, i.e.,

$$r^*_{j_i} = \frac{c_{j_i} + \beta q_{j_i}}{1-\alpha} \qquad \text{for } i = 1, ..., m. \tag{3.20}$$

We use this result to develop a faste algorithm for the case of linear $\eta(\theta)$.

*Proposition* 3.4.12. Suppose $\eta(\theta) = \alpha\theta + \beta$, where $\alpha \leq 0$ and $\beta$ are constants. Solving (4.2) is equivalent to solving (3.5) with selling prices $r_j = \frac{c_j + \beta q_j}{1-\alpha}$ for $j = 1, ..., n$.

By Proposition 3.4.12, the problem with endogenous prices can be solved using the algorithm for the exogenous prices case when the inverse hazard rate function is linear. This result is useful because our solution method for the exogenous case is generally faster than that for the endogenous case (despite the fact that they have the same theoretical complexity). This is because the set of nodes in *Shortest Path 'Exo'* is generally smaller than the set of nodes in *Shortest Path 'Endo'*.

In the special case where $K = 0$ and $\eta(\theta)$ is linear, the following proposition provides an another method to obtain the optimal solution.

*Proposition* 3.4.13. When $K = 0$ and $\eta(\theta) = \alpha\theta + \beta$, where $\alpha \leq 0$ and $\beta$ are constants, $j \in S^*$ if and only if there exists at least two values of $\theta \in [\underline{\theta}, \overline{\theta}]$ such that $\theta q_j - r_j = \max_{i=1,...,n}(\theta q_i - r_i)$ where $r_i = \frac{c_i + \beta q_i}{1-\alpha}$ for $i = 1, ..., n$.

Proposition 3.4.13 shows when $K = 0$ and $\eta(\theta)$ is linear, the optimal assortment can also be obtained graphically by looking for the upper envelope of the utility curves drawn in the positive quadrant for $\theta \in [\underline{\theta}, \overline{\theta}]$ for each product using prices $r_j = \frac{c_j + \beta q_j}{1-\alpha}$ for $j = 1, ..., n$ as shown in the following example. Note that this method has the same complexity as the *Zero Fixed Cost Algorithm*. The following example illustrates how it works.

*Example* 3.3. (Cont'd from Examples 3.1 and 3.2) Let $\vec{c}$, $\vec{q}$ and the customer valuation distribution $F(\theta)$ be as in Examples 3.1 and 3.2. We have $\eta(\theta) = \frac{1-\theta}{b}$ so the inverse hazard rate is linear in $\theta$ (with $\alpha = -\frac{1}{b}$ and $\beta = \frac{1}{b}$). When $b = 0$, we obtain \$17.5, \$20.25 and \$75 respectively for the prices of products 1, 2 and

3. In Figure 3.5, each line corresponds to the utility that customers get from each product when they are offered at these prices. The upper envelope of the three lines in the positive quadrant only intersects the lines from products 2 and 3, therefore $\{2, 3\}$ is the optimal assortment.

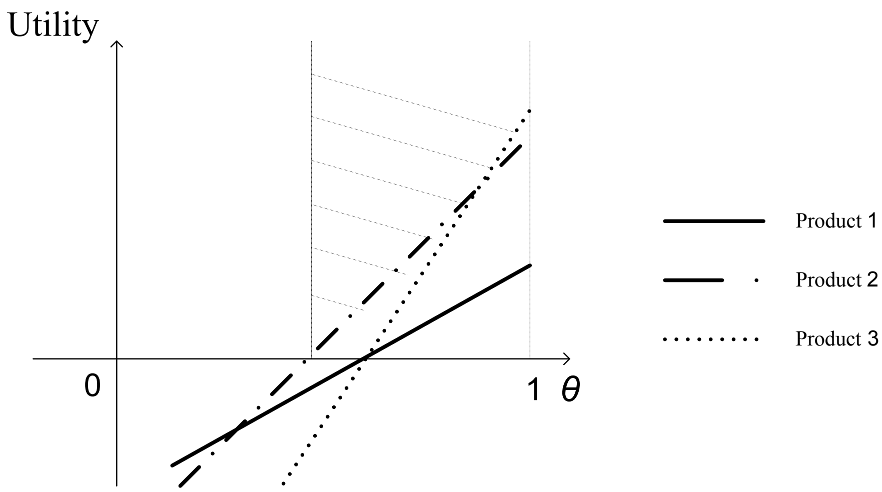


FIGURE 3.5: Upper envelope graph for Example 3.3.

In summary, we have developed a number of efficient algorithms to identify the optimal assortment when prices are endogenous. Table 3.6 provides a summary of our solution methods as a function of $K$ and $\eta(\theta)$. Note that all of the solution methods of complexity can be prefaced by the elimination of dominated products from the set of products to consider.

## 3.5 Discussion and Insights

In Sections 3.3 and 3.4 we show how to obtain the optimal solution efficiently for the exogenous and endogenous prices settings. In this section, we

| | Non-linear inverse rate function $\eta(\theta)$ | Linear inverse rate function $\eta(\theta)$ |
|---|---|---|
| $K > 0$ | *Shortest Path 'Endo'* $O(n^3)$ | use *Shortest Path 'Exo'* with (3.20) $O(n^3)$ |
| $K = 0$ | *Zero Fixed Cost Algorithm* $O(n^2)$ | Upper envelope with (3.20) $O(n^2)$ |

Table 3.6: Solution methods to identify the optimal assortment.

discuss some interesting properties of these solutions and compare the two cases.

We have shown that the optimal assortment in the exogenous prices case may contain dominated products while the optimal assortment in the endogenous prices case does not. This is because the retailer who sets prices is able to increase the expected profit by pricing dominated products high enough so that no customer buys them. Including a product with zero purchase probability in the assortment does not increase the profit, therefore, the optimal assortment does not contain dominated products. In contrast, the retailer who does not set prices may want to include dominated products because products prices are generally sub-optimal (in the sense that they are different from the prices that she would choose if she could).

Another interesting observation is about how the distribution of customer valuations affects the optimal assortment. When the fixed cost is negligible (i.e., $K = 0$) and prices are endogenous, the optimal assortment does not vary with the distribution of customer valuations (as long as it is IFR). On the contrary, the optimal assortment may vary with the distribution of customer valuations in the exogenous prices case. The underlying reason is as follows. When prices are decision variables, the conditions of Lemma 3.4.6 guarantee that there exists a set of prices such that each product in $S^*$ makes a positive contribution to the expected profit. Because the retailer has the ability to set prices, these conditions are sufficient to guarantee the optimality of assortment. However, in the exogenous case, the retailer has to work with prices which are often suboptimal so the contribution of a product to the expected profit can become negative as the distribution of customer valuation changes. Therefore, the optimal assortment might change with the distribution of customer valuation.

Note that, in the endogenous prices case, the retailer who only has incomplete information about the distribution of customer valuations can still identify the optimal assortment as long as the distribution is known to be an IFR distribution. However, setting optimal prices requires knowledge of the distribution.

We use Example 3.4 below to further analyze how the optimal assortments in the two cases differ.

*Example* 3.4. A retailer can choose from two vertically differentiated products with $\vec{c} = (1, 0.5)$ and $\vec{q} = (20, 40)$. The distribution is $F(\theta) = 1 - (1 - \theta)^b$ with support $[0, 1]$, $b = 6$ and $K = 0$.

When the prices are fixed with $\vec{r} = (2, 19)$, it is optimal to offer both products. However when the retailer is free to set prices, it is optimal to offer only product 2 at a price of \$6.56.

This example shows that the optimal assortment in the exogenous prices case can contain two products $i$ and $j$ such that $c_i > c_j$ and $q_i < q_j$, that is, one product has a higher quality and a lower price than another product. By Lemma 3.4.1, this is not true in the endogenous price case. Table 3.7 summarizes the properties of the optimal assortment in both cases in terms of cost, quality levels, and selling prices in both settings.

| Condition† | Exogenous prices case | Endogenous prices case |
|---|---|---|
| $\hat{r}_{j_1} < ... < \hat{r}_{j_m}$ | true | true |
| $\frac{\hat{r}_{j_1}}{q_{j_1}} < ... < \frac{\hat{r}_{j_1}}{q_{j_1}}$ | true | true |
| $\hat{r}_{j_1} - c_{j_1} < ... < \hat{r}_{j_m} - c_{j_m}$ | not necessarily true‡ | true |
| $\frac{c_{j_1}}{q_{j_1}} < ... < \frac{c_{j_m}}{q_{j_m}}$ | not necessarily true | true |
| $c_{j_1} < ... < c_{j_m}$ | not necessarily true | true |
| $\frac{\hat{r}_{j_1}}{q_{j_1}} < \frac{\hat{r}_{j_2}-r_{j_1}}{q_{j_2}-q_{j_1}} < ... < \frac{\hat{r}_{j_m}-\hat{r}_{j_{m-1}}}{q_{j_m}-q_{j_{m-1}}} < \bar{\theta}$ | true | true |
| $\frac{c_{j_1}}{c_{j_1}} < \frac{c_{j_2}-c_{j_1}}{q_{j_2}-q_{j_1}} < ... < \frac{c_{j_m}-c_{j_{m-1}}}{q_{j_m}-q_{j_{m-1}}} < \bar{\theta}$ | not necessarily true | true |

† $\hat{r}_j = r_j$ in the exogenous prices case and $\hat{r}_j = r^*_j$ in the endogenous prices case.

‡ Because can hold as an equality in the exogenous prices case.

TABLE 3.7: Comparison of the properties of the optimal assortment in exogenous and endogenous prices case.

From the comparison of the optimal assortments in Examples 3.1 and 3.2, along with Example 3.4, we find that the optimal assortment in the exogenous prices case can be larger or smaller than the optimal assortment in the endogenous prices case and that one is not necessarily a subset of the other. In particular, a product which is included in the optimal assortment when prices are fixed may be dropped when the retailer is able to set her own prices. It is therefore necessary for a retailer who acquires the freedom of setting selling prices to re-evaluate her whole assortment and re-optimize using the appropriate algorithm.

Example 3.5 illustrates some interesting properties of the optimal assortment as the set of candidate products to choose from shrinks.

*Example* 3.5. A retailer can choose from three vertically differentiated products with $\vec{c} = (2, 6, 10)$ and $\vec{q} = (10, 14, 20)$. The distribution of customer valuations is uniform over $[0, 1]$ and $K = 0$.

The optimal assortment when prices are endogenous is $S^* = \{1, 3\}$ and optimal prices are $r_1^* = 6$ and $r_3^* = 15$.

Now suppose that product 1 is no longer available, that is, the retailer can only offer a subset of $\{2, 3\}$. In that case, the optimal assortment is $\{2, 3\}$ and the optimal prices are $r_2^* = 10$ and $r_3^* = 15$.

From this example, we find that, if a product from the optimal assortment becomes unavailable, for example, because the manufacturer discontinues its production or the retailer stock outs of it and the manufacturer is not able to refill the inventory in a timely manner, then it may be optimal for the retailer to include new products in the assortment. This result provides a possible explanation for why some manufacturers would continue to offer products which are not included in the optimal assortment when the retailers can choose from the full list of potential products (another possible explanation is that the manufacturer also sells his products to other retailers operating in different markets and for whom the optimal assortment is different because their cost structure is different). Note that this property is also true in the exogenous prices case. To see this, consider the counterpart of example 3.5 where prices are exogenous and the price vector $\vec{r} = (6, 10, 15)$ and notice that the optimal assortments are the same.

However Proposition 3.5.1 shows that, when prices are endogenous and $K = 0$, it is never optimal to drop a product which is still available from the optimal assortment.

*Proposition* 3.5.1. Suppose $K = 0$. Let $S^*_{\mathcal{N}}$ be the optimal assortment when the set of potential products to choose from is $\mathcal{N}$. Let $S^*_{\mathcal{N}'}$ denote the optimal assortment when the set of potential products to choose from is $\mathcal{N}' \subseteq \mathcal{N}$. We have $(S^*_{\mathcal{N}} \cap \mathcal{N}') \subseteq S^*_{\mathcal{N}'}$.

## 3.6 Conclusion

We study the problem of a retailer offering an assortment of vertically differentiated products to customers who differ in their valuation of quality. We first consider the scenario where prices are exogenously determined and the retailer's only decision is to decide the set of products to offer. We show that the problem of finding the optimal assortment can be modeled as a shortest path problem, which has complexity $O(n^3)$. Interestingly we show that the optimal assortment may contain dominated products, i.e., products with has a lower quality, lower selling price, and higher cost than one other product.

Second, we examine the setting where the retailer can also determine the selling prices of products which are included in the assortment. We show that this problem can also be modeled as a shortest path with complexity $O(n^3)$. However, in practice, this problem usually takes longer to solve because the network generally contains more nodes and arcs. When the fixed cost associated with each offered product is negligible (i.e., $K = 0$), we develop a more efficient algorithm, with complexity $O(n^2)$. If the distribution of customer valuations has a linear inverse hazard rate we show that the optimal prices can be obtained independently of the optimal assortment so that the optimal assortment can be found by using the solution method developed for the exogenous prices case.

Barghava & Choudhary (2001) consider the problem of selecting and pricing vertically differentiated products in the absence of fixed cost but only give a partial characterization of the optimal solution. In contrast, we provide efficient methods to obtain the optimal assortment for any quality levels and variables cost, in the presence of fixed cost and also consider the exogenous prices case. Further, we provide a number of interesting insights and guidelines to practitioners regarding their product mix strategy.

Nowadays, most retailers still do not use the tools of assortment planning to manage and price their products. For instance, Amazon currently determines the selling prices of its products separately, not as a category. As a result there could be some inconsistencies in the selling prices. In our future work, we plan to analyze data from online retailers and look for opportunities to improve their profitability using our model.

There are a number of interesting extensions to our work. First, we assume that the products differ with respect to only one attribute which can be regarded a combination of a product's many characteristics. We are interested in explicitly considering multiple attributes to capture the complexity of consumer choice. Second, we do not consider the case in which the retailer may have constraints when determining the selling prices of products. For example, the retailer does not want to set a price that is higher than a competitor's selling price. Third, our model assumes that the retailer is a monopolist and that the product quality levels and variable costs are fixed and determined by the manufacturer(s). Interesting extensions would be to consider a setting in which the manufacturer(s) choose(s) the product quality levels and transfer costs in anticipation of the retailer's assortment choice or a setting in which two or more retailers compete in prices when selling the same manufacturers' products. Finally, our model does not include the presence of inventory and therefore does not incorporate the impact of stock-outs and the resulting substitution behavior of customers. We plan to explore these extensions in our future research.

# Chapter 4

# Optimal Pricing and Bundling of Vertically Differentiated Information Goods

## 4.1 Introduction

Bundling has become an increasingly popular pricing strategy for sellers to increase their sales and profits. We can easily find examples for a wide range of products, for instance, telecommunication products (e.g., internet, phone, and TV), food (e.g., bags of potato chips of different flavors packed into one big box), vacation packages (e.g., airline ticket, hotel, and transportation including shuttle or rent-a-car), and software (e.g., Microsoft Office including Microsoft Word, Excel, Access, and Powerpoint). The iTunes store is the world's leading online music and TV shows store with over 10 million songs and over 30,000 TV episodes for sale. When buying on iTunes, consumers can choose to buy songs and TV episdoes individually or in a bundle, in the form of an album or a complete TV season.

Bundling can be classified into two categories: price bundling and product bundling (Stermersch and Tellis 2002). Price bundling is the sale of two or more separate products in a package at a discount without any integration of the products while product bundling is the integration and sale of two or more separate products or services at any price. Sellers use three strategies for price and product bundling: (1) no bundling, where each product is sold individually; (2) pure bundling, where products are sold only in a package; and (3) mixed bundling, where products are sold individually and in a bundle. Motivated by the successful practice of the iTunes, in this essay, we study the assortment planning problem with price bundling and mixed bundling strategy. For ease of exposition, we define a product sold individually as a *component*. Songs can be ordered based on their popularity which is displayed on iTunes, so we assume that they are *vertically* differentiated; consequently, if all the songs have the same price, we assume that all customers would prefer a more popular song to a less popular one. We model the customer choice using a widely applied pure characteristic demand model (Bhargava and Choudhary 2001). We address the following questions: which individual components should the firm offer? Should the firm offer a bundle containing all the components? And how should the firm price the offered individual components and the bundle in order to maximize its profit?

## 4.2 Literature Review

As the popularity of bundling has grown in practice, so has academic research on bundling from economics, marketing and operational perspective. Stremersch & Tellis (2002) provide a comprehensive summary of early research on bundling in economics and marketing literature. The economists start with the work Stigler (1963) and study whether the bundling is beneficial to the sellers or the consumers. Stigler uses a vector of reservation prices to represent customer demand information. Customers choose the product maximizing their surplus, i.e., difference between their reservation price and the product price. The result shows that bundling can increase the seller's profit for two negatively correlated goods. Utilizing Stigler's framework, Adams & Yellen (1976) consider more than two customer segments and show that mixed bundling is typically optimal. Schmalensee (1984) assumes that reservation prices follow a bivariate normal distribution and uses numerical analysis to show that mixed bundling is a favorable strategy especially for negative correlated products. McAfee *et al.* (1989) identify the conditions under which mixed bundling is preferred for two goods. Salinger (1995) uses a graphical analysis to demonstrate that the benefit of bundling relates to the correlation of reservation prices among products, demand function and cost of products. Venkatesh & Kamakura (2003) study the optimal bundling strategies for complements and substitute goods. All these studies focus on the case of two products.

Due to the complexity in analyzing bundles, few general results are available for bundles of more than two goods. Bakos & Brynjolfsson (1999) and Geng *et al.* (2005) examine the optimality condition of the pure bundling strategy for a large numbers of information goods with negligible marginal cost. Chuang & Sirbu (1999)) show that selling all individual goods and the bundle of all goods dominates pure bundling or no bundling for academic journals.

Another stream of research focus on customer behavior, such as how consumers evaluate product bundles, see for example Yadav & Monroe (1993), Yadav

(1994), Chung & Rao (2003), Jedidi *et al.* (2003). Also, several studies use optimization approaches to design and price the bundle. Hanson & Martin (1990) investigate how to determine the optimal bundle prices for firms to maximize the profit given the number of customers and their reservation price in each segment. They formulate the problem as a mixed integer linear model and find the profit maximizing bundle prices without considering the entire set of feasible solutions. Bitran & Ferrer (2007) address the problem of how to determine the profit maximizing composition and the price of a bundle meeting specified constraints in a competitive environment. They formulate the problem as a nonlinear mixed integer problem and propose a solution approach to determine the optimal composition and price of the bundle. Hitt & Chen (2005) and Wu *et al.* (2008) study the customized bundle pricing strategy for information goods using non-linear programming approach. In the customized bundling strategy, customers choose a certain number of goods out of a large pool of different goods, and pay the price dependent only on the number of goods in the bundle regardless of the specific content in the bundle.

As the above discussion indicates, past work has mainly focused on bundling of two goods and few available results are for general bundles of more than two goods. To our best knowledge, our work is the first one to work on the mixed bundling strategy for vertically differentiated information goods.

## 4.3 Model

Suppose a firm sells $n$ vertically differentiated components and a bundle containing all $n$ components.. Let $q_j$ denote the quality level of component $j$. Without loss of generality, we assume that $q_1 \geq q_2 \geq ... \geq q_n \geq 0$. The marginal cost of component $j$ is $c_j$ and component $j$ is sold at price $r_j$. For notational convenience, suppose we have a fictitious product 0 with $q_0 = c_0 = 0$. Let $\vec{c} = (c_1, ..., c_j)$ and $\vec{r} = (r_1, ..., r_j)$.

Let $Q_{n+1}$, $C_{n+1}$ and $R_{n+1}$ respectively denote the quality level, variable cost and selling price of the bundle containing all $n$ components. We assume that the quality level of the bundle is $Q_{n+1} = q_1 + q_2 + ... + q_n + q_{n+1}$ where $q_{n+1} \geq 0$ captures the extra benefit to the customer (convenience, packaging, etc.) of buying all products into a bundle rather than one by one. We assume the marginal cost of the bundle is additive, i.e, $C_{n+1} = c_1 + ... + c_n$. For ease of exposition, let $c_{n+1} = 0$.

Customers can either buy one component, multiple components or the bundle. We assume that customers are characterized by their willingness to pay for one unit of quality in the product category, or *valuation*, which is measured by $\theta$. A customer with valuation $\theta$ gets utility $\theta q_j - r_j$ from buying component $j$ and $\theta Q_{n+1} - R_{n+1}$ from buying the bundle. We assume that utilities are additive, i.e., the utility of buying components 1 and 2 is $\theta(q_1 + q_2) - (r_1 + r_2)$ for a customer with valuation parameter $\theta$. This is a standard assumption in price bundling since the components are usually consumed separately. Also, we assume that customers do not get a positive utility from buying more than one unit of the same component or bundle. Without loss of generality, we assume that the utility of not buying anything is equal to zero.

Let $F(\theta)$ denote the distribution of customers' valuations. Let $h(\theta) = \frac{f(\theta)}{1-F(\theta)}$ be the failure rate of $F$ and $\xi(\theta) = \theta h(\theta)$ be its generalized failure rate. $F(\theta)$ is an *increasing failure rate* (IFR) distribution if $h(\theta)$ is weakly increasing for all $\theta$. $F(\theta)$ is an *increasing generalized failure rate* (IGFR) distribution if $\xi(\theta)$ is weakly increasing for all $\theta$ as defined in Lariviere & Porteus (2001). Note that if a distribution is IFR it is also IGFR, but the reverse does not hold since many IGFR distributions are not IFR. Most common distributions are IGFR distributions, e.g., uniform, normal, logistic, chi-squared, exponential, Laplace, and Weibull distributions. We use $\eta(\theta) = \frac{1}{h(\theta)}$ to denote the inverse failure rate function. If $F(\theta)$ is an IFR distribution, then $\eta'(\theta) < 0$. In what follows, we assume that $F$ is an IFR distribution unless otherwise specified.

In the example of iTunes, the "quality level" of a song/or series episode can be estimated by the customers through customer and critical reviews, as well as the "popularity" index which measures the relative frequency with which each song or episode has been downloaded by previous users. The valuation $\theta$ of a customer represents the degree of familiarity with and attachment to a particular artist's work (in the case of music). For example, a fan of a particular rock artist is characterized by a very high value of $\theta$ for the music of that artist, while a customer who in general despises rock music will have a very low valuation $\theta$. Since all the songs of a particular artist are usually of a similar genre, it is reasonable to assume that a customer has the same valuation for all the 10 to

15 songs in an album. Similarly, all the episodes of a show are of the same kind (e.g., comedy, drama, reality...), hence it is reasonable to assume that a customer has the same valuation for all the episodes of a season.

Let $\theta_j = \frac{r_j}{q_j}$ be the valuation of the customer who gets zero utility from buying component $j$. We assume that the firm decides prices such that

$$\underline{\theta} \leq \theta_1 \leq ... \leq \theta_n \leq \bar{\theta}. \tag{4.1}$$

It follows that every customer with valuation parameter $\theta > \theta_j$ gets a positive utility from buying component $j$ but also a positive utility from buying components 1 to $j-1$. Hence, in the absence of the bundle, a customer with taste parameter $\theta \in [\theta_j, \theta_{j+1})$ buys components 1 to $j$. Such a customer buys the bundle if and only if $\theta Q_{n+1} - R_{n+1} \geq \theta(q_1 + ... + q_j) - (r_1 + ... + r_j)$ and $\theta Q_{n+1} - R_{n+1} \geq 0$. Let $\theta_{n+1} = \frac{R_{n+1} - (r_1 + ... + r_n)}{q_{n+1}}$ be the lowest value of $\theta$ such that a customer buys the bundle.

For given values of $\vec{r}$ and $R_{n+1}$, customers with $\theta \geq \theta_{n+1}$ buy the bundle. Customers with $\theta < \theta_{n+1}$ and $\theta \in [\theta_j, \theta_{j+1})$ buy components 1 to $j$. Finally, customers with $\theta < \theta_{n+1}$ and $\theta < \theta_1$ do not buy anything. Let $P_{n+1}$ be the proportion of customers who buy the bundle, $P_j$ be the proportion of customers who buy components 1 to $j$ and $P_0$ be the proportion of customer who do not buy anything, we have:

$$
\begin{aligned}
P_{n+1} &= 1 - F(\theta_{n+1}) \\
P_j &= F(\theta_{j+1}) - F(\theta_j) \\
P_0 &= \begin{cases} F(\theta_1) & \text{if } \theta_{n+1} > \theta_1 \\ F(\theta_{n+1}) & \text{if } \theta_{n+1} > \theta_1 \end{cases}
\end{aligned}
$$

We can write the profit function as follows:

$$
\begin{aligned}
\mathbb{E}\Pi(R_{n+1}, \vec{r}) &= (1 - F(\theta_{n+1}))(R_{n+1} - C_{n+1}) \\
&\quad + \sum_{i=1}^{n-1} (F(\theta_{i+1}) - F(\theta_i))(r_1 + ... + r_i - (c_1 + ... + c_i))
\end{aligned}
$$

The firm's objective is to find $R_{n+1}, \vec{r}$ such that

$$
\mathbb{E}\Pi(R^*_{n+1}, \vec{r}^*) = \max_{R_{n+1}, \vec{r}} \mathbb{E}\Pi(\vec{r}).
$$

This model is a special case of the model for the general vertically differentiated products in Chapter 3, where we consider a product category of vertically differentiated products in which each customer buys at most one product. To see this, let product $j$ be composed of components 1 to $j$, for $j = 1, ..., n$. The variable cost of product $j$ is $C_j \equiv c_1 + ... + c_j$, its quality level is $Q_j \equiv q_1 + ... + q_j$ and its selling price is $R_j \equiv r_1 + ... + r_j$. Also let product $n+1$ correspond to the bundle, with variable cost $C_{n+1}$, quality level $Q_{n+1}$ and selling price $R_{n+1}$. Each customer buys at most one unit of product $j$, for $j = 1, ...n+1$. We exploit this analogy between the two models in Section 4.4. Note that in Sections 4.5.2 and 4.5.4, our model is no longer equivalent to the model for the general vertically differentiated products. Moreover, we provide more results about conditions under which offering the bundle is optimal and more insights under special cases.

Note that there is a one-to-one correspondence between $\vec{r}, R_{n+1}$ and $\vec{\theta} = (\theta_1, ..., \theta_{n+1})$. Let $\vec{\theta}$ be the vector corresponding to $\vec{r}, R_{n+1}$, then we can rewrite the expected function as a function of $\vec{\theta}$ only as:

$$\mathbb{E}\Pi(\vec{\theta}) = \sum_{i=1}^{n+1}(1 - F(\theta_i))(\theta_i q_i - c_i)$$

The firm's profit maximization problem is

$$\begin{array}{l} \max_{\vec{\theta}} \mathbb{E}\Pi(\vec{\theta}) \\ s.t. \;\; \underline{\theta} \leq \theta_1 \leq \theta_2 \leq ... \leq \theta_n \leq \theta_{n+1} \leq \bar{\theta} \end{array} \tag{4.2}$$

## 4.4 Results

Without loss of generality we can assume that $\theta_n \leq \theta_{n+1} \leq \overline{\theta}$. This condition along with (4.1), allow us to divide the components into two groups. For $j = 1, ..., n$, we either have $\theta_j = \theta_{j+1}$ or $\theta_j < \theta_{j+1}$. In the first case, no customer buys components 1 to $j$ only, while, in the second, some customers buy components 1 to $j$. Similarly for the bundle, we can either have $\theta_{n+1} = \overline{\theta}$ or $\theta_{n+1} < \overline{\theta}$. In the first case, no customer buys the bundle, while in the second, some customers buy the bundle. Given $\theta_j$ for $j = 1, ..., n+1$, let $S = \{j = 1, ..., n+1 : \theta_j < \theta_{j+1} \text{ and } \theta_j < \bar{\theta}\}$.

The firm's objective is to find the optimal set of components and/or the bundle to offer and charge optimal prices for the offered products. A naive way is to enumerate all the possible combination of components and the bundle, which is very time consuming. We aim to develop efficient methods to identify the optimal set and characterize the optimal prices for offered products. We first establish some nice properties about the optimal set $S$ defined above. Let the optimal set $S^* = \{j_1, ..., j_m\}$ such that $j_1 < j_2 < ... < j_m$. We denote $j_0 = 0$.

*Lemma* 4.4.1. $S^*$ is optimal if and only if the costs and quality levels satisfy

$$\frac{c_1 + ... + c_{j_1}}{q_1 + ... + q_{j_1}} < \frac{c_{j_1+1} + ... + c_{j_2}}{q_{j_1+1} + ... + q_{j_2}} < ... < \frac{c_{j_{m-1}+1} + ... + c_{j_m}}{q_{j_{m-1}+1} + ... + q_{j_m}} < \bar{\theta}, \tag{4.3}$$

and for $i = 0, ..., m-1$

$$\begin{cases} \frac{c_{\{j_i+1\}}+...+c_k}{q_{\{j_i+1\}}+...+q_k} > \frac{c_{k+1}+...+c_{j_{i+1}}}{q_{k+1}+...+q_{j_{i+1}}} & \text{for } k = j_i + 1, ..., j_{i+1} - 1 \\ \frac{c_{\{j_m+1\}}+...+c_k}{q_{\{j_m+1\}}+...+q_k} > \bar{\theta} & \text{for } k > j_m \end{cases} \tag{4.4}$$

Furthermore, $S^*$ is unique.

Lemma 4.4.1 shows that the ratio of the incremental cost to the incremental quality for each product in the optimal set is strictly increasing in the quality level.

Suppose we identify the optimal set $S^*$, Lemma 4.4.2 provides a method to compute the optimal prices $r^*$.

*Lemma* 4.4.2. The values of $\theta^*_{j_i}$ for $i = 1, ..., m$ are obtained by solving:

$$\theta^*_{j_i} \quad = \quad \eta(\theta^*_{j_i}) + \frac{c_{j_{i-1}+1} + ... + c_{j_i}}{q_{j_{i-1}+1} + ... + q_{j_i}} \qquad \text{for } i = 1, ..., m. \tag{4.5}$$

where $j_0 = 0$. For $k \notin S^*$, we have $\theta^*_k = \theta^*_{k+1}$ if $k < j_m$ and $\theta^*_k = \bar{\theta}$ if $k > j_m$. The optimal prices are obtained as follows:

$$\begin{array}{lll} r^*_{j_i} & = & \theta^*_{j_i} q_{j_i} & \text{for } i = 1, ..., m-1, \\ r^*_{j_m} & = & \theta^*_{j_m} q_{j_m} & \text{if } j_m < n+1, \\ R^*_{j_m} & = & r^*_{j_1} + ... + r^*_{j_{m-1}} + \theta^*_{j_m}(q_{j_{m-1}+1} + q_{j_m}) & \text{if } j_m = n+1. \end{array}$$

Bundling is a prevalent sales strategy. We next characterize the necessary and sufficient condition for adopting the bundling strategy.

*Lemma* 4.4.3. The optimal set $S^*$ includes the bundle, i.e., $n+1 \in S^*$, if and only if

$$\frac{c_i + ... + c_n}{q_i + ... + q_n + q_{n+1}} < \overline{\theta} \qquad \text{for } i = 1, ..., n. \tag{4.6}$$

Barghava & Choudhary (2001) only prove that (4.6) is a sufficient condition. We show that it is also a necessary condition. Also Lemma 4.4.3 shows that the decision whether or not to include the bundle does not depend on the distribution $F$ (as long as it is IFR).

Firms also adopt the pure bundling strategy at times. Proposition 4.4.4 provides the conditions under which the optimal set only includes the bundle.

*Proposition* 4.4.4. The pure bundling strategy is optimal, i.e., $S^* = \{n+1\}$ if and only if

$$\frac{c_1 + ... + c_n}{q_1 + ... + q_n + q_{n+1}} \leq \frac{c_1 + ... + c_i}{q_1 + ... + q_i} \qquad \text{for } i = 1, ..., n \tag{4.7}$$

Intuitively, pure bundling is optimal if and only if it is more efficient to manufacture the bundle, i.e, if the cost-quality ratio of the bundle is the lowest.

When the conditions of Proposition 4.4.4 are not satisfied, finding the optimal solution can be done through an exhausting search over all possible sets $S^*$. Using Lemma 4.4.1, we can show that it is never optimal to have $n$ and $n+1 \in S^*$; hence, the number of sets $S$ to consider is $2^n - 1$, which is still time consuming.

Notice that the contribution to the expected profit of component $j_i$ in the set $S$ depends only on the adjacent components/products $j_{i-1}$ and $j_{i+1}$ since other products do not have impact on the purchase probability of product $j_i$. This property enables us to model this problem as a shortest path problem. Furthermore, the adjacent products have an impact on the contribution of product $j_i$, therefore, we need to use a pair of product as a node in the graph. The arc cost is proportional to the expected profit of a product. Using the properties of Lemma 4.4.1, we are able to construct a parsimonious network as follows.

**ALGORITHM: Shortest Path Algorithm**

- Step 1. Construct node set: $V$ consists of the following nodes

$$\begin{aligned} V \quad = \quad & \left\{(0,i) : i = 1, ..., n+1 \text{ and } \frac{c_1 + ... + c_i}{q_1 + ... + q_i} < \bar{\theta}\right\} \\ & \cup \{(0,0), (n+2, n+2)\} \\ & \cup \{(j, n+2) : j = 1, ..., n+1 \text{ and } \frac{c_1 + ... + c_j}{q_1 + ... + q_j} < \bar{\theta}\} \\ & \cup \left\{(i,j) : 1 \le i < j \le n+1 \text{ and } \frac{c_{i+1} + ... + c_j}{q_{i+1} + ... + q_j} < \bar{\theta}\right\}, \end{aligned}$$

where $(0,0)$ and $(n+2,n+2)$ are fictitious starting and ending node respectively.

- Step 2. Construct arc set: Add an arc from node $(i,j)$ to $(l,k) \in V$ to set $A$ if $j = l = k$ or $j = l < k$ and

$$\begin{cases} \frac{c_{i+1}+...+c_j}{q_{i+1}+...+q_j} < \frac{c_{j+1}+...+c_k}{q_{j+1}+...+q_k} < \bar{\theta} & \text{if } k < n+2 \\ \frac{c_{i+1}+...+c_j}{q_{i+1}+...+q_j} < \bar{\theta} & \text{if } k = n+2 \text{ and } l < k \end{cases}$$

- Step 3. Compute arc costs:

$$c_{(i,j),(j,k)} = \begin{cases} -\left[r_{j+1} + .... + r_k - (c_{j+1} + ... + c_k)\right]\left[1 - F(\theta_k)\right] & \text{if } k < n+2 \\ 0 & \text{if } k = n+2 \end{cases}$$

where $\theta_k$ is the solution to $\theta_k = \eta(\theta_k) + \frac{c_{\{j+1\}}+...+c_k}{q_{\{j+1\}}+...+q_k}$ and $r_l = \theta_k q_l$ for $l = j+1, ..., k$.

- Step 4. Solve the shortest path problem from $(0,0)$ to $(n+2,n+2)$.

*Theorem* 4.4.5. The problem of finding an optimal assortment reduces to a shortest path problem between $(0,0)$ and $(n+2,n+2)$.

*Corollary* 4.4.6. The complexity of the method to find an optimal expected profit is $O(n^3)$.

Using the *Zero Fixed Cost Algorithm* stated in Section 3.4.3 of Chapter 3, we develop a more efficient algorithm to identify the optimal bundling strategy.

**Algorithm: Backward Search Algorithm**

- Step 0. $S^* = \emptyset$, $i = 0$.

- Step 1. If $i < n+1$,

  For $j := n+1$ down to $i+1$,

  If $\frac{c_{i+1}+...+c_j}{q_{i+1}+...+q_j} < \overline{\theta}$ and $\frac{c_{i+1}+...+c_k}{q_{i+1}+...+q_k} > \frac{c_{k+1}+...+c_j}{q_{k+1}+...+q_j}$

  for $k = i+1, ..., j-1$

  $S^* := S^* \cup \{j\}$, $i := j$ and back to step 1.

  end

  end

  end

- Step 2: Use Lemma 4.4.2 to obtain $\vec{r}^*$ and/or $R^*_{n+1}$.

*Proposition* 4.4.7. The set $S$ obtained from the Algorithm is optimal.

*Example* 4.1. A firm can choose from six components with $\vec{q} = [5, 4.3, 3.3, 2.5, 1.0, 0.8]$ and $\vec{c} = [3, 2.5, 1.6, 1.5, 1, 0.5]$ and one bundle containing all the components with $q_{n+1} = 0.5$. she knows customer valuations follow a uniform distribution with support $[0, 1]$. The firm decides what products of offer and how much to charge on each offered product in order to maximize the profit.

Using the backward search algorithm, we get the optimal set $S^* = \{1, 2, 3, 4, 7\}$, and the retailer sets the price such that some customers purchase the first 3 or 4 components or the bundle. From Lemma 4.4.2, we obtain the optimal values of $\theta$ and prices are as follows.

| | $\theta$ | $r^*$ | included in $S^*$? |
|---|---|---|---|
| component 1 | 0.78 | 3.91 | Yes |
| component 2 | 0.78 | 3.36 | Yes |
| component 3 | 0.78 | 2.58 | Yes |
| component 4 | 0.80 | 2.00 | Yes |
| component 5 | 0.83 | / | No |
| component 6 | 0.83 | / | No |
| bundle | 0.83 | 13.75 | Yes |

TABLE 4.1: Optimal $\theta$ and prices in Example 4.1.

*Proposition* 4.4.8. The optimal assortment $S^*$ includes all the components in the absence of the bundle, i.e., $S^* = \{1, 2, ..., n\}$, if

$$\frac{c_1}{q_1} < \frac{c_2}{q_2} < ... < \frac{c_n}{q_n} < \bar{\theta}. \tag{4.8}$$

In the presence of the bundle, if

$$\frac{c_i + ... + c_n}{q_i + ... + q_n + q_{n+1}} < \overline{\theta} \qquad \text{for } i = 1, ..., n. \tag{4.9}$$

the optimal assortment $S^{*'} = \{1, ..., j, n+1\}$ where $j$ is smallest integer such that

$$\frac{c_{j+1}}{q_{j+1}} > \frac{c_{j+2} + ... + c_n}{q_{j+2} + ... + q_{n+1}}.$$

Moreover, the prices for components 1 to $j$ in the presence of the bundle are the same as those in the absence of the bundle.

Proposition 4.4.8 reveals that if a firm's best strategy is to offer all the components in the absence of the bundle, and she wants to introduce bundling strategy later, she only needs to determine the best optimal price $R^*_{n+1}$ without changing the prices for the components.

## 4.5 Special Cases

In practice, the cost functions are special under some situations; for instance, iTunes distribute each song at an approximate marginal cost of \$0.50 according to Wu *et al.* (2008); and the advanced technology makes even possible to distribute information goods at negligible marginal cost. The firm may charge the same price for all components following the specific industry tradition. For example, ITunes used to charge \$0.99 for each individual song. We next exploit properties under different conditions.

### 4.5.1 Same Positive Marginal Cost and Different Component Price

For many practical examples (such as iTunes) the variable cost is the same for each component, that is, $c_1 = ... = c_n = c$. In this case we are able to obtain full characterization of the optimal solution.

*Proposition* 4.5.1. Assume that $q_1 > q_2 > ... > q_n$.

(1) If

$$\frac{(n-i+1)c}{q_i + ... + q_n + q_{n+1}} < \overline{\theta} \qquad \text{for } i = 1, ..., n \tag{4.10}$$

(1.a) If there exists a positive integer $0 < j < n$ satisfying

$$q_j > \frac{q_{j+1} + ... + q_{n+1}}{n-j} \tag{4.11}$$

then $S^* = \{1, ..., \hat{j}, n+1\}$ where $\hat{j}$ be the largest integer $j$ satisfying condition (4.11)

(1.b) If there is no positive integer satisfying condition (4.11), i.e.,

$$q_1 < \frac{q_{j+1} + ... + q_{n+1}}{n-1}, \tag{4.12}$$

then the pure bundling strategy is optimal, i.e., $S^* = \{n+1\}$.

(2) If condition (4.10) is not satisfied, then $S^* = \{1, ..., \tilde{j}\}$ where $\tilde{j}$ is the largest integer $j$ such that $\frac{c_j}{q_j} < \bar{\theta}$.

Condition (4.10) determines whether or not the bundle should be included in the optimal assortment. For components, we compare the quality of a component with the average quality of the components with lower quality level. The optimal set includes all components whose quality is higher than the average quality of the components with lower quality level.

#### 4.5.1.1 Example and insights

In this section we assume that the distribution of customer valuations has a a beta distribution[1] with $a = 1$, i.e., the probability density function is $f(\theta) = b(1-\theta)^{b-1}$ and cumulative distribution function is $F(\theta) = 1 - (1-\theta)^b$ with $\theta \in [0, 1]$.

[1]The probability density function is $B(\theta; a, b) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{\beta(a,b)}$, where $\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$.

Suppose condition (4.10) is satisfied, then from Proposition 4.5.1, we have $S^* = \{1, ..., \hat{j}, n+1\}$ where $\hat{j}$ be the largest integer $j$ satisfying condition (4.11). Let $\hat{c}_j = \frac{c}{q_j}$ for $j \leq \hat{j}$ and $\hat{c}_{n+1} = \frac{(n-\hat{j})c}{q_{\hat{j}+1}+...+q_{n+1}}$. We have $\theta_j^* = \frac{\hat{c}_j b+1}{b+1}$, $r_j^* = \theta_j^* q_j$ for $j \leq \hat{j}$ and $R_{n+1}^* = r_1^* + ... + r_{\hat{j}}^* + \theta_{n+1}(q_{\hat{j}+1} + ... + q_{n+1})$.

Next, we perform a sensitivity analysis with respect to $c$ and $b$.

- ***Cost and Purchasing Levels***. Customers with $\theta < \theta_1^*$ purchase nothing and customers with $\theta > \theta_{n+1}^*$ purchase the bundle.

$$\begin{aligned} \frac{\partial \theta_{n+1}^*}{\partial c} &= \frac{(n-\hat{j})b}{(q_{\hat{j}+1} + ... + q_{n+1})(b+1)} > 0 \\ \frac{\partial \theta_1^*}{\partial c} &= \frac{b}{q_1(b+1)} > 0 \end{aligned}$$

  An increase in the cost $c$ results in a strict increase in $\theta_1$ and $\theta_{n+1}$. Therefore, as cost increases, fewer customers purchase the bundle and the proportion of customers making a purchase decreases too.

- ***Market Evolution***. The distribution of customer valuations changes with the value of $b$. For a new product early-stage market, more consumers are low type ones, i.e., having low $\theta$. As the market matures, the distribution of customers over purchasing level evens out and more consumers are high type ones. Sundararajan (2004) argues that a gradual decrease in $b$ reflects the market evolution.

$$\begin{aligned}
\frac{\partial\theta^*_{n+1}}{\partial b} &= -\frac{\eta(\theta^*_{n+1})}{b+1} < 0 \\
\frac{\partial\theta^*_{1}}{\partial b} &= -\frac{\eta(\theta^*_{1})}{b+1} < 0
\end{aligned}$$

The proportion of consumers buying the bundle is

$$\begin{aligned}
P_{n+1} &= (1-\theta_{n+1})^b \\
\frac{\partial P_{n+1}}{\partial b} &= (1-\theta_{n+1})^b ln(1-\theta_{n+1}) + b(1-\theta_{n+1})^{b-1}\frac{\eta(\theta^*_{n+1})}{b+1} \\
&= (1-\theta_{n+1})^b \left[ ln(1-\theta_{n+1}) + \frac{1}{b+1} \right] \\
&= (1-\theta_{n+1})^b \left[ ln\left(\frac{b(1-\hat{c}_{n+1})}{b+1}\right) + \frac{1}{b+1} \right]
\end{aligned}$$

Let $\omega(b) = \left[ ln\left(\frac{b(1-\hat{c}_{n+1})}{b+1}\right) + \frac{1}{b+1} \right]$, then

$$\frac{\partial\omega(b)}{\partial b} = \frac{1}{b(b+1)^2} > 0$$

and $\omega(\infty) < 0$. Hence $\frac{\partial P_{n+1}}{\partial b} < 0$. The proportion of customers who do not make purchase is

$$P_0 = 1 - (1-\theta_1)^b = -(1-\theta_1)^b \left[ ln\left(\frac{b(1-\hat{c}_1)}{b+1}\right) + \frac{1}{b+1} \right] > 0.$$

This result shows that in early-stage market, the proportion of customers buying nothing is high and a low bundling price is a good strategy to penetrate the market. As the market matures, the firm increase the price of the bundle and the total market share increases.

- ***Profits***. The expected profit is as follows.

$$\begin{aligned}
\mathbb{E}\Pi^* &= \sum_{i=1}^{\hat{j}}[1-F(\theta_i^*)](r_i^*-c) \\
&\quad +[1-F(\theta_{n+1}^*)]\left[r_{\hat{j}+1}^*+...+r_{n+1}^*-(n-\hat{j})c\right] \\
\frac{\partial \mathbb{E}\Pi^*}{\partial c} &= -\sum_{i=1}^{\hat{j}}\left[[1-F(\theta_i^*)]+f(\theta_i^*)(r_i^*-c)\frac{b}{q_i(b+1)}\right]-n[1-F(\theta_{n+1})] \\
&\quad -f(\theta_{n+1})\left[r_{\hat{j}+1}^*+...+r_{n+1}^*-(n-\hat{j})c\right]\frac{(n-\hat{j})b}{(q_{\hat{j}+1}+...+q_{n+1})(b+1)} \\
&= -\sum_{i=1}^{\hat{j}}\left[[1-F(\theta_i^*)]+f(\theta_i^*)\eta(\theta_i^*)\frac{b}{b+1}\right] \\
&\quad -n[1-F(\theta_{n+1})]-f(\theta_{n+1})\eta(\theta_{n+1}^*)\frac{(n-\hat{j})b}{(b+1)} \\
&= -\sum_{i=1}^{\hat{j}}[1-F(\theta_i^*)]\frac{2b+1}{b+1}-\left[[1-F(\theta_{n+1})]\frac{n(b+1)+(n-\hat{j})b}{b+1}\right]<0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbb{E}\Pi^*}{\partial b} &= -\sum_{i=1}^{\hat{j}}f(\theta_i^*)\frac{\eta(\theta_i^*)}{b+1}(r_i^*-c) \\
&\quad -f(\theta_{n+1}^*)\frac{\eta(\theta_{n+1}^*)}{b+1}\left[r_{\hat{j}+1}^*+...+r_{n+1}^*-(n-\hat{j})c\right] \\
&= -\sum_{i=1}^{\hat{j}}\frac{[1-F(\theta_i^*)](r_i^*-c)}{b+1} \\
&\quad -\frac{[1-F(\theta_{n+1}^*)]\left[r_{\hat{j}+1}^*+...+r_{n+1}^*-(n-\hat{j})c\right]}{b+1}<0
\end{aligned}$$

Profits are strictly decreasing with $c$ and $b$.

### 4.5.2 Same Positive Marginal Cost and Same Component Price

Some retailers, such as iTunes, charge the same price for each component. Under this situation, we have $c_1 = ... = c_n = c$, $r_1 = ... = r_n = r$. Let $t_j$ be the bundle price that a customer is indifferent buying between components from 1 to $j$ and the bundle, where

$$t_j \;=\; jr + r\frac{q_{j+1} + .... + q_n + q_{n+1}}{q_j} \qquad \text{for } j = 1, ..., n.$$

The expected profit is

$$\mathbb{E}\Pi(R_{n+1}, r) \;=\; \begin{cases} \left[1 - F\left(\theta_{n+1}\right)\right]\left(R_{n+1} - nc\right) & \text{if } R_{n+1} \leq t_1 \\ \left[1 - F\left(\theta_{n+1}\right)\right]\left[R_{n+1} - jr - c(n-j)\right] & \\ + \sum_{i=1}^{j}[1 - F(\theta_i)](r - c) & \\ \qquad \text{if } t_j < R_{n+1} \leq t_{j+1}, j = 1, ..., n-1 & \\ \sum_{i=1}^{n}[1 - F(\theta_i)](r - c) & \text{if } t_n < R_{n+1} \end{cases}$$

Note that the expected profit function is continuous in $R_{n+1}$:

$$\begin{aligned} \mathbb{E}\Pi(t_j^+) \;&=\; \left[1 - F\left(\theta_j\right)\right]\left(t_j - nc\right) + \left[F\left(\theta_j\right) - F\left(\theta_j\right)\right] j(r - c) \\ &\quad + \sum_{i=1}^{j-1}\left[F\left(\theta_{i+1}\right) - F\left(\theta_i\right)\right] i(r - c) \\ &= \mathbb{E}\Pi(t_j^-), \end{aligned}$$

and the first derivative is.

$$\frac{\partial \mathbb{E}\Pi}{\partial R_{n+1}} \;=\; \begin{cases} 1 - F(\theta_{n+1}) - \frac{f(\theta_{n+1})(R_{n+1} - nc)}{Q_{n+1}} & \text{if } R_{n+1} \leq t_1 \\ 1 - F(\theta_{n+1}) - \frac{f(\theta_{n+1})[R_{n+1} - jr - (n-j)c]}{Q_{n+1} - Q_j} & \\ \qquad \text{if } t_j < R_{n+1} \leq t_{j+1}, j = 1, ..., n & \\ 0 & \text{if } t_n < R_{n+1} \end{cases}$$

However, the first derivative is not continuous since

$$
\begin{aligned}
\left.\frac{\partial \mathbb{E}\Pi}{\partial R_{n+1}}\right|_{R_{n+1}=t_j^-} &= 1 - F(\theta_{n+1}) - \frac{f(\theta_{n+1})[R_{n+1} - jr - (n-j)c]}{Q_{n+1} - Q_j} \\
&\neq \left.\frac{\partial \mathbb{E}\Pi}{\partial R_{n+1}}\right|_{R_{n+1}=t_j^+} \\
&= 1 - F(\theta_{n+1}) - \frac{f(\theta_{n+1})[R_{n+1} - (j-1)r - (n-j+1)c]}{Q_{n+1} - Q_{j-1}}
\end{aligned}
$$

The expected profit at $t_j$ has a jump up if

$$
\begin{aligned}
& \frac{q_j}{Q_{n+1} - Q_j} > \frac{r - c}{R_{n+1} - jr - (n-j)c} \\
\Leftrightarrow \quad & \frac{n-j}{Q_{n+1} - Q_j} > \frac{1}{q_j} \\
\Leftrightarrow \quad & \frac{Q_{n+1} - Q_j}{n-j} < q_j
\end{aligned}
$$

Hence, there could be multiple stationary points and the global maximum can be found at $t_j$ for some $j = 1, ..., n$.

*Proposition* 4.5.2. For a given $r$, if

$$
\frac{Q_{n+1} - Q_j}{n-j} > q_j \qquad \text{for } j = 1, ..., n-1 \tag{4.13}
$$

then the FOC is

$$
\theta_{n+1} = \eta(\theta_{n+1}) + \frac{(n-j)c}{Q_{n+1} - Q_j} \qquad \text{for } j = 0, ..., n-1 \tag{4.14}
$$

and we get the optimal $R^*_{n+1}$ by comparing the $\mathbb{E}\Pi(R_{n+1})$ at all points satisfying condition 4.14.

When both $r$ and $R_{n+1}$ are decision variables, either mixed bundling strategy or pure bundling strategy could be optimal.

### 4.5.3 Zero Marginal Cost and Different Component Price

Advances in information technology have greatly reduced the distribution cost of information goods. It is now possible to sell information goods with negligible marginal cost, i.e. $c_1 = ... = c_n = 0$. Bakos & Brynjolfsson (1999) and Geng *et al.* (2005) assume that information goods have zero marginal cost.

In this case, (4.7) is satisfied and we show that pure bundling strategy is always optimal.

*Proposition* 4.5.3. A firm, who sells information goods with negligible costs, always uses pure bundling strategy.

### 4.5.4 Zero Marginal Cost and Same Component Price

In this section, we look at the special case when $c_1 = ... = c_n = 0$ and $r_1 = ... = r_n = r$.

*Proposition* 4.5.4. Assume the component price $r$ is fixed. If $F$ is a continuous IGFR distribution, then $\theta^*_{n+1}$ is the unique solution to:

$$1 - F(\theta_{n+1}) - \theta_{n+1} f(\theta_{n+1}) = 0. \tag{4.15}$$

Let $g(\theta) = 1 - F(\theta) - \theta f(\theta)$, and $j^*$ be the largest number of components that customers may purchase. For IGFR continuous distribution, we have

$$
\begin{aligned}
g(\theta^*_{n+1}) &= 0 \\
P^*_{n+1} &= 1 - F(\theta^*_{n+1}) \\
P^*_j &= \begin{cases} F(\theta_j) - F(\theta_{j-1}) & \text{if } j < j^* \\ F(\theta^*_{n+1}) - F(\theta_j) & \text{if } j = j^* \\ 0 & \text{if } j > j^* \end{cases} \\
R^*_{n+1} &= \begin{cases} Q_{n+1}\theta^*_{n+1} & \text{if } \theta^*_{n+1} \le \theta_1 \\ jr + (Q_{n+1} - Q_j)\theta^*_{n+1} & \text{if } \theta_j < \theta^*_{n+1} \le \theta_{j+1} \\ nr + q_{n+1}\theta^*_{n+1} & \text{if } \theta_n < \theta^*_{n+1} \end{cases} \\
\mathbb{E}\Pi^* &= \begin{cases} (1 - F(\theta^*_{n+1}))Q_{n+1}\theta^*_{n+1} & \text{if } \theta^*_{n+1} \le \theta_1 \\ (1 - F(\theta^*_{n+1}))(Q_{n+1} - Q_j)\theta^*_{n+1} & \\ + \sum_{i=1}^{j}(1 - F(\theta_i))r & \text{if } \theta_j < \theta^*_{n+1} \le \theta_{j+1} \\ (1 - F(\theta^*_{n+1}))q_{n+1}\theta^*_{n+1} + \sum_{i=1}^{n}(1 - F(\theta_i))r & \text{if } \theta_n < \theta^*_{n+1} \end{cases}
\end{aligned}
$$

*Proposition* 4.5.5. (a) $\mathbb{E}\Pi^*$ is increasing in $q_{n+1}$ and $q_j$ for $j = 1, ..., n$, non-decreasing in $r$.

(b) $R^*_{n+1}$ is increasing in $q_{n+1}$ and $q_j$ for $j > j^*$, non-decreasing in $r$.

(c) $j^*$ is non-decreasing in $q_j$ and non-increasing in $r$.

*Example* 4.2. Let $F(\theta) = 1-(1-\theta)^b$. Note that the uniform distribution on $[0, 1]$ is a special case of this distribution obtained by setting $b = 1$. If $b > 1$, then the distribution function is concave, meaning that there are more customers with low taste parameters. If $0 < b < 1$, then the distribution function is convex, meaning that there are more customers with high taste parameters. The generalized failure rate is $\xi(\theta) = \frac{b\theta}{1-\theta}$, so $F$ is IGFR.

In this case, (4.15) becomes $(1-(b+1)\theta_{n+1})\left(1-\theta_{n+1}\right)^{b-1} = 0$ so that $\theta^*_{n+1} = \frac{1}{b+1}$ and $R^*_{n+1} = j^*r + \frac{Q_{n+1}-Q_{j^*}}{b+1}$ where $j^*$ is the lowest integer such that $\theta_{j^*+1} \geq \frac{1}{b+1}$. Also we have

$$
\begin{aligned}
P^*_{n+1} &= \left(\frac{b}{b+1}\right)^b \\
P^*_j &= \begin{cases} (1-\theta_j)^b - (1-\theta_{b+1})^b & \text{if } j < j^* \\ (1-\theta_{j^*})^b - \left(\frac{b}{b+1}\right)^b & \text{if } j = j^* \\ 0 & \text{if } j > j^* \end{cases}
\end{aligned}
$$

*Proposition* 4.5.6. When both $r$ and $R_{n+1}$ are decision variables and $F$ is a continuous IGFR distribution, pure bundling is optimal.

*Proposition* 4.5.7. When both $r$ and $R_{n+1}$ are decision variables, a mixed bundling strategy such that some customers purchase the first component and some customers purchase the bundle cannot be optimal for any distribution $F$.

*Example* 4.3. (Continuing from Example 4.2) When $F(\theta) = 1-(1-\theta)^b$, it is optimal to set $\theta^*_{n+1} = \frac{1}{b+1}$, $R^*_{n+1} = \frac{Q_{n+1}}{b+1}$ and $r \geq \frac{q_1}{b+1}$. As a result, $P_{n+1} = \left(\frac{1}{b+1}\right)^b$ and $P_j = 0$ for $j = 1, ..., n$, i.e., no customer buys individual components. The optimal expected profit is equal to $\left(\frac{b}{b+1}\right)^b \frac{Q_{n+1}}{b+1}$.

## 4.6 Conclusion

We study the bundling problem of a retailer offering vertically differentiated information goods. Although all customers prefer more to less for the quality level, they value the quality level differently. We characterize the conditions under which including the bundle is optimal, the conditions under which a pure bundling strategy is optimal, and the conditions under which a mixed bundling strategy is optimal. We develop efficient methods, i.e., Shortest Path Algorithm and Backward Search Algorithm with complexity $O(n^3)$ and $O(n^2)$ respectively, to identify what components and/or the bundle to offer and address how to optimally price the components and/or the bundle. Interestingly, we find that, under some situations, introducing the bundle does not affect the prices of components.

We study a number of special cases, such as the case of identical marginal cost for all components (positive or zero) and the case in which all components are sold at the same price. For these special cases, we provide valuable insights. For instance, in the case of the same positive marginal cost and different component price, we fully characterize the optimal bundling strategy. When the distribution of customer valuations follows a special beta distribution, we show that a low price for the bundle is a good strategy to penetrate a new product market where customers have low valuations of quality. As the market evolutes, the bundling prices increases and the total market share increases as well. When the marginal cost is zero, our results show that a pure bundling strategy is optimal, which is consistent with results in literature.

In this work, we only consider one bundle containing all the components. One interesting extension of this work is to consider multiple bundles as iTunes currently offers basic and "deluxe" bundles. Moreover, in practice, some customers may buy one component, then update their valuation about the quality level then buy some more. We are interested in examining this problem in the presence of customer valuation update.

# Chapter 5

# Conclusion

In this dissertation, we aim to develop a valuable decision support model for an integrated sourcing and delivery optimization problem and provide retailers insights and guidelines about their marketing strategy. In chapter 1, we develop an integrated sourcing and delivery optimization model to minimize the total procurement, transportation, and vehicle assignment cost. To obtain the near-optimal solution within short time, we propose effective solution methodologies. This approach, currently in use at a leading company to support their material supply and delivery planning activities, yields significant savings of millions of dollars annually. Moreover, this model is a valuable support tool for tactical planning, capable of performing what-if analysis to provide useful managerial insights.

For the assortment planning problem for vertically differentiated products, in Chapter 3, we develop, efficient methods to identify the optimal assortment to offer in the exogenous prices case, and to identify the optimal assortment to offer and the optimal price to charge for each offered product in the endogenous prices case. Our findings reveal that dominated products might be included in the optimal assortment and the optimal assortment may vary with the distribution of customer valuations in the exogenous prices case. In contrast, in the endogenous prices case, the optimal assortment never includes dominated products. Moreover, when the fixed cost is negligible, the optimal assortment does not vary with the distribution of customer variations. Therefore, even when we have incomplete information about customer valuations, we are able to identify the optimal assortment.

For the problem of optimal pricing and bundling of vertically differentiated information goods, we address how to choose the optimal bundling strategy. We provide conditions under which introducing the bundle is desirable, conditions under which pure bundling is optimal, and conditions under which mixed bundling is optimal. We develop efficient methods to identify the set of components and/or the bundle to offer and how much to charge for each offered component and/or the bundle. In the special cases when variable costs and/or the component prices are the same, we provide valuable insights such as how the bundling strategy and purchasing probability of products change with the evolution of the market for a new product.

# Appendices

# Appendix A

# Supplemental Material for Chapter 2

*Notation*

- Indices/Sets
  - $Q$: set of suppliers
  - $J$: set of customers
  - $V$: set of vehicle types
  - $T$: set of base time periods
  - $Q(j)$: subset of suppliers that can supply customer $j \in J$
  - $R(j)$: subset of suppliers to which vehicles can return after a delivery to customer $j \in J$
  - $J(q)$: subset of customers that supplier $q \in Q$ can serve $q$
  - $TJ(j)$: set of feasible delivery periods for customer $j \in J$ in terms of base periods
  - $TQ(q)$: set of open periods for supplier $q \in Q$ in terms of base periods
  - $T(v)$: set of time periods for vehicle type $v$

- Parameters

  - $D_j$: number of deliveries of customer $j$
  - $B_j$: average quantity per delivery of customer $j$
  - $C_j^v$: vehicle assignment cost per delivery to customer $j$ by vehicle of type $v$
  - $P_q$: unit price of supplier $q$
  - $I_q$: initial inventory of supplier $q$
  - $M_q$: maximum supply capacity over the planning horizon of supplier $q$
  - $G_{qt}$: production quantity in period $t$ of supplier $q$
  - $N^v$: number of vehicles of type $v$ available
  - $F_{qj}$: full-load transportation cost ($ per mile) from supplier $q$ to customer $j$
  - $E_{jq}$: empty-load transportation cost ($ per mile) from customer $j$ to supplier $q$

- Decision Variables

  - $x_{qjt}^v$ number of vehicles of type $v$ that deliver material from supplier $q$ to customer $j$ in period $t$, for all $v \in V, j \in J, q \in Q(j), t \in TQ(q) \cap TJ(j) \cap T(v)$

- $y_{jqt}^{v}$ number of vehicles of type $v$ that go to supplier $q$ after delivering material to customer $j$ in period $t$, for all $v \in V, j \in J, q \in R(j), t \in TJ(j) \cap T(v), t+1 \in TQ(q) \cap T(v)$
- $r_{qt}^{v}$ number of vehicles of type $v$ that enter the system at supplier $q$ at period $t$, for all $v \in V, q \in Q, t \in TQ(q) \cap T(v)$
- $s_{jt}^{v}$ number of vehicles of type $v$ that leave the system after delivering to customer $j$ in period $t$, for all $v \in V, j \in J, t \in TJ(j) \cap T(v)$

*Proof of Proposition 2.3.1.* Consider any integer solution that a vehicle delivers one full-load of material to customer $j$ with unit demand from supplier $q$ in time period $t$; the left-hand-side (LHS) of constraint (2.8) $x_{qjt}$ equals 1. Since customer $j$ has unit demand, no delivery is made to customer $j$ before time period $t$, and thus no return trip from customer $j$ to supplier $q$, i.e. $y_{jq(t-\tau_{jq})} = 0$. For the flow conservation constraint (2.5) at supplier $q$ in time period $t$, the LHS $r_{qt} + \sum_{j \in J} y_{jq(t-\tau_{jq})}$ equals $r_{qt} + \sum_{j' \neq j} y_{j'q(t-\tau_{j'q})}$ , and the right hand side (RHS) $\sum_{j \in J} x_{qjt}$ is not smaller than $x_{qjt}$. So, $r_{qt} + \sum_{j' \neq j} y_{j'q(t-\tau_{j'q})} \geq x_{qjt}$ . For other time periods, the LHS of constraint (2.8) $x_{qjt}$ equals 0, and thus the inequality is valid. □

*Proof of Proposition 2.3.2.* Consider any integer solution that $m(\geq 1)$ vehicles delivery material to customers in subset $J'$ from the subset of suppliers $Q'$ in time interval $[t_1, t_2]$. For supplier subset $Q'$, time interval $[t_1, t_2]$, and customer subset $J'$, the LHS of constraint (2.10) is at least $m$ since those $m$ vehicles delivering material in the subsystem (subset $Q'$ and subset $J'$) enter the subsystem at supplier subset $Q'$ from outside the system, or at suppliers in subset $Q'$ from customers, or at customers in subset $J'$ from suppliers within the time interval. The right-hand side of constraint (2.10) is at most $m$ since $\sum_{t=t_1}^{t_2} \sum_{q \in Q'} \frac{x_{qj^*t}}{D_{j^*}} \leq 1$. Hence, the given solution satisfies this inequality. $\square$

*Proof of Proposition 2.3.3.* We prove the propositißon by checking two cases: the minimum number of vehicles needed to satisfy the demand $K$ is greater than the number of vehicles serving the subsystem and $K$ is no more than the number of vehicles serving the subsystem.

- Case 1. The minimum number of vehicles needed to satisfy the demand is greater than the number of vehicles serving the subsystem $z$, i.e., $K > z$ . Let $U(t) = \left\lceil \frac{t_2 - t}{\tau_{Q'J'}} \right\rceil$ and $L(t) = \left\lfloor \frac{t_2 - t}{\tau_{Q'J'}} \right\rfloor$ represent the maximum number of trips and minimum number of trips that a vehicle can finish in time interval $[t, t_2]$, respectively. The total available vehicle capacity is greater than the number of deliveries that vehicles can make, so we have

$$\sum_{q\in Q'}\sum_{t=t_1}^{t_2}U(t)r_{qt}+\sum_{j\in J'}\sum_{q\in Q'}\sum_{t=t_1-\tau_{jq}}^{min(t_1-1,t_2-\tau_{jq})}U(t+\tau_{jq})y_{jqt}$$
$$+\sum_{j\notin J'}\sum_{q\in Q'}\sum_{t=t_1-\tau_{jq}}^{t_2-\tau_{jq}}U(t+\tau_{jq})y_{jqt}$$
$$+\sum_{q\in Q'}\sum_{j\in J'}\sum_{t=t_1-\tau_{qj}}^{min(t_1-1,t_2-\tau_{qj}-\tau_{jQ'})}U(t+\tau_{qj}+\tau_{jQ'})x_{qjt}$$
$$+\sum_{q\notin Q'}\sum_{j\in J'}\sum_{t=t_1-\tau_{qj}}^{t_2-\tau_{qj}-\tau_{jQ'}}U(t+\tau_{qj}+\tau_{jQ'})x_{qjt}$$
$$-\sum_{j\in J'}\sum_{t=t_1}^{t_2-\tau_{jQ'}}L(t)s_{jt}-\sum_{j\in J'}\sum_{q\notin Q'}\sum_{t=t_1}^{t_2-\tau_{jQ'}}L(t)y_{jqt}-\sum_{q\in Q'}\sum_{j\notin J'}\sum_{t=t_1}^{t_2-\tau_{jQ'}}L(t)x_{qjt}$$
$$\geq\sum_{t=t_1}^{t_2}\sum_{j\in J''}\sum_{q\in Q'}x_{qjt}. \tag{A.1}$$

Therefore, inequality (2.12) is valid.

- Case 2. The minimum number of vehicles needed to satisfy the demand is greater than the number of vehicles serving the subsystem, i.e.,$K \leq z$ .

  - Case 2.1. $RK - z'$ is no greater than zero. The validity of inequality (2.12) is obvious since

$$\sum_{t=t_1}^{t_2}\sum_{j\in J''}\sum_{q\in Q'}x_{qjt}\leq W$$

– Case 2.2. $RK - z'$ is greater than zero . So $RK > z'$, i.e.,

$$
\begin{aligned}
K \quad > \quad & \sum_{q\in Q'}\sum_{t=t_1}^{t_1+\lambda-1} r_{qt} + \sum_{j\in J'}\sum_{q\in Q'}\sum_{t=t_1-\tau_{jq}}^{min(t_1-1,t_1+\lambda-1)} y_{jqt} \\
& + \sum_{j\notin J'}\sum_{q\in Q'}\sum_{t=t_1-\tau_{jq}}^{t_1+\lambda-1} y_{jqt} + \sum_{q\in Q'}\sum_{j\in J'}\sum_{t=t_1-\tau_{jq}}^{min(t_1-1,t_1+\lambda-1)} x_{qjt} \\
& + \sum_{q\notin Q'}\sum_{j\in J'}\sum_{t=t_1-\tau_{qj}}^{t_1+\lambda-1} x_{qjt} - \sum_{j\in J'}\sum_{t=t_1}^{t_1+R-1} s_{jt} \\
& - \sum_{j\in J'}\sum_{q\notin Q'}\sum_{t=t_1}^{t_1+R-1} y_{jqt} - \sum_{q\in Q'}\sum_{j\notin J'}\sum_{t=t_1}^{t_1+R-1} x_{qjt}.
\end{aligned}
$$

So, $W - (RK - z')$ is greater than the LHS of (A.1). Therefore, inequality (2.12) is valid.

$\square$

*Separation Procedure for Residual Capacity Inequality*

Let key set $KJ$ denote the collection of $J''$, set $KJ = \emptyset$.

For each job $j^* \in J$ with due period $t_{j^*}$, set the inequality violation value $\Delta_{j^*} = 0$, identify the delivery time window $[\bar{t}_1, \bar{t}_2]$ in the LP solution, i.e., $\sum_{q\in Q}\sum_{t=1}^{\bar{t}_1-1} \bar{x}_{qj^*t} = 0, \sum_{q\in Q} \bar{x}_{qj^*\bar{t}_1} > 0, \sum_{q\in Q}\sum_{t=\bar{t}_1}^{\bar{t}_2} \bar{x}_{qj^*t} = D_j$;

- Step 1. Sort the periods with positive forward flows to customer $j^*$ in the decreasing order of $\sum_{q\in Q} \bar{x}_{qj^*t}$. Let $m$ denote the number of periods with positive flows to customer $j^*$, and $T' = \{t^{(1)}, t^{(2)}, ., t^{(m)}\}$ denote the sorted set of $m$ periods, i.e., $\sum_{q\in Q} \bar{x}_{qj^*t^{(1)}} \geq \sum_{q\in Q} \bar{x}_{qj^*t^{(2)}} \geq ... \geq \sum_{q\in Q} \bar{x}_{qj^*t^{(m)}}$. Set the time interval $[t_1, t_2] = [t^{(1)}, t^{(1)}]$.

- Step 2. For all $t' = t^{(2)}, ., t^{(m)}$, if $t' \notin [t_1, t_2]$,

  - Step 2.1. set $J' = J'' = \{j^*\}$, $Q' = \emptyset$; if $t_2 < t'$, set time interval $[t_1, t_2] = [t_1, t']$; if $t' < t_1$, set time interval $[t_1, t_2] = [t', t_2]$; Add supplier $q$ to $Q'$ if $\bar{x}_{qj^*t_2} > 0$;

  - Step 2.2. for $t = t_2 - 1$ to $t_1$, construct $Q', J'$as follows.

    * *Customer subset $J'$ construction.* For all $j$ not in $J'$ with $\sum_{q \in Q'} \bar{y}_{jqt} > 0$, identify the suppliers $\bar{Q}(j)$ serving the customer $j$ in the time interval in the current LP solution (i.e., all suppliers $q$ with $\sum_{t=t_1}^{t_2-1} \bar{x}_{qjt} > 0$), and add $j$ to $J'$ if

    $$\sum_{t=t_1}^{t_2-1} \sum_{q \in Q'} \bar{y}_{jqt} > \sum_{j' \in J} \sum_{q \in \bar{Q}(j) \setminus Q'} \bar{y}_{j'q(t'_1-1)} + \sum_{t=t_1}^{t_2-1} \sum_{q \in \bar{Q}(j) \setminus Q'} \bar{r}_{qt} > 0$$

    * *Supplier subset $Q'$ construction.* For all $q$ not in $Q'$ with $\sum_{j \in J'} \bar{x}_{qjt} > 0$ , add $q$ to $Q'$ if

    $$\sum_{t=t_1}^{t_2-1} \sum_{j \in J'} \bar{x}_{qjt} > \sum_{j \in J} \bar{y}_{jq(t_1-1)} + \sum_{t=t_1}^{t_2-1} \bar{r}_{qt}$$

  - Step 2.3. *Customer subset $J'$ augmentation.* For all $j$ not in $J'$, add customer $j$ to $J'$ if

    $$\sum_{t=t_1}^{t_2-1} \sum_{q \in Q'} \bar{y}_{jqt} > \sum_{t=t_1}^{t_2-1} \sum_{q \notin Q'} \bar{x}_{qjt}$$

  - Step 2.4. *Supplier subset $Q'$ augmentation.* For all $q$ not in $Q'$, add supplier $q$ to $Q'$ if

$$\sum_{t=t_1}^{t_2-1}\sum_{j\in J'}\bar{x}_{qjt} > \sum_{j\in J}\bar{y}_{jq(t_1-1)} + \sum_{t=t_1}^{t_2-1}\sum_{j\notin J'}\bar{y}_{jqt} + \sum_{t=t_1}^{t_2-1}\bar{r}_{qt}$$

– Step 2.5. Compute the number of vehicles ($z$) in the system and the minimum number of vehicles ($K$) needed to satisfy the total demand ($W$) of customers in $J''$.

$$\begin{aligned} z &= \sum_{q\in Q'}\sum_{t=t_1}^{t_2}\bar{r}_{qt} + \sum_{j\in J'}\sum_{q\in Q'}\bar{y}_{jq(t_1-1)} + \sum_{j\notin J'}\sum_{q\in Q'}\sum_{t=t_1-1}^{t_2-1}\bar{y}_{jqt} \\ &\quad + \sum_{q\notin Q'}\sum_{j\in J'}\sum_{t=t_1}^{t_2-1}\bar{x}_{qjt} - \sum_{q\in Q'}\sum_{j\notin J'}\bar{x}_{qjt_1} \\ K &= \left\lceil \frac{W}{t_2-t_1+1} \right\rceil \end{aligned}$$

If $z$ is fractional, go to step 2.6; otherwise, next $t$. if $K > \lceil z \rceil$, next $t$. Define the proportion of satisfied demand in time interval $[t_1, t_2]$ as $\sum_{q\in Q'}\sum_{t=t_1}^{t_2}\bar{x}_{qjt}/D_j$.

– Step 2.6. For all $j$ in $J'$ not in $J''$ with fully met demand in the time interval $[t_1, t_2]$ , i.e., the proportion of satisfied demand equals one, add $j$ to $J''$.

Recompute the minimum number of vehicles ($K$) needed to satisfy the total demand ($W$) of customers in $J''$. If $K < z$, go to step 2.7; if $K > \lceil z \rceil$, next $t$; otherwise, go to step 2.8.

– Step 2.7. Sort the customers in $J'$ not in $J''$ in the decreasing order of the proportion of satisfied demand in the time interval, and add the customers in the sorted subset to subset $J''$ until the minimum number of vehicles needed in order to meet the demand of customers in $J''$ is $\lceil z \rceil$, i.e., $K = \lceil z \rceil$. If $K > \lceil z \rceil$, next $t$.

– Step 2.8. Compute the inequality violation value

$$
\begin{aligned}
\Delta \;=\;& W - \sum_{q\in Q'}\sum_{j\in J'}\sum_{t=t_1}^{t_2} x_{qjt} + RK \\
& - \sum_{q\in Q'}\sum_{t=t_1}^{t=t_2} R'(t) r_{qt} + \sum_{j\in J'}\sum_{q\in Q'} \sum_{t=t_1-\tau_{jq}}^{min(t_1-1,t_2-\tau_{jq})} R'(t+\tau_{jq}) y_{jqt} \\
& + \sum_{j\notin J'}\sum_{q\in Q'} \sum_{t=t_1-\tau_{jq}}^{t_2-\tau_{jq}} R'(t+\tau_{jq}) y_{jqt} \\
& + \sum_{q\in Q'}\sum_{j\in J'} \sum_{t=t_1-\tau_{qj}}^{min(t_1-1,t_2-\tau_{qj}-\tau_{jQ'})} R'(t+\tau_{qj}+\tau_{jQ'}) x_{qjt} \\
& + \sum_{q\notin Q'}\sum_{j\in J'} \sum_{t=t_1-\tau_{qj}}^{t_2-\tau_{qj}-\tau_{jQ'}} R'(t+\tau_{qj}+\tau_{jQ'}) x_{qjt} - \sum_{j\in J'} \sum_{t=t_1}^{t_2-\tau_{jQ'}} R''(t) s_{jt} \\
& - \sum_{j\in J'}\sum_{q\notin Q'} \sum_{t=t_1}^{t_2-\tau_{jQ'}} R''(t) y_{jqt} - \sum_{q\notin Q'}\sum_{j\notin J'} \sum_{t=t_1}^{t_2-\tau_{jQ'}} R''(t) x_{qjt}.
\end{aligned}
$$

If $\Delta < \Delta_{j^*}$, record $[t_1, t_2], Q', J', J''$.

Next $t'$;

- Step 3. For $J''$ with the smallest $\Delta_{j^*}(< 0)$, if $J''$ is not in the key set $KJ$, set the inequality violation value for $J''$, $\Delta_{j''} = \Delta_{j^*}$, and record the associated $Q', J', [t_1, t_2]$; else, if $\Delta_{j^*} < \Delta_{j''}$, set $\Delta_{j''} = \Delta_{j^*}$, and update $Q', J', [t_1, t_2]$ corresponding to $J''$ .

For each subset $J''$ in the key set $KJ$, add the residual capacity inequalities for the corresponding $Q', J', [t_1, t_2]$.

# Appendix B

# Supplemental Material for Chapter 3

*Proof of Lemma* 3.3.1. Notice that (3.6) and (3.7) follow directly from (3.1). We prove (3.8) by contradiction. Let $k$ be the smallest integer such that $r_{j_k} - c_{j_k} > r_{j_{k+1}} - c_{j_{k+1}}$. There are two cases: (1) $k = m-1$ or $r_{j_k} - c_{j_k} > r_{j_i} - c_{j_i}$ for $i = k+1, ..., m$, or (2) otherwise.

In case (1), let $S = \{j_1, ..., j_k\}$. We have

$$\begin{aligned}
\mathbb{E}\Pi(S) - \mathbb{E}\Pi(S^*) &= \left[1 - F\left(\frac{r_{j_k} - r_{j_{k-1}}}{q_{j_k} - q_{j_{k-1}}}\right)\right](r_{j_k} - c_{j_k}), \\
&\quad - \sum_{i=k}^{m-1}\left[F\left(\frac{r_{j_{i+1}} - r_{j_i}}{q_{j_{i+1}} - q_{j_i}}\right) - F\left(\frac{r_{j_i} - r_{j_{i-1}}}{q_{j_i} - q_{j_{i-1}}}\right)\right](r_{j_i} - c_{j_i}) \\
&\quad - \left[1 - F\left(\frac{r_{j_m} - r_{j_{m-1}}}{q_{j_m} - q_{j_{m-1}}}\right)\right](r_{j_m} - c_{j_m}) \\
&> \left[1 - F\left(\frac{r_{j_k} - r_{j_{k-1}}}{q_{j_k} - q_{j_{k-1}}}\right)\right](r_{j_k} - c_{j_k}) \\
&\quad - \sum_{i=k}^{m-1}\left[F\left(\frac{r_{j_{i+1}} - r_{j_i}}{q_{j_{i+1}} - q_{j_i}}\right) - F\left(\frac{r_{j_i} - r_{j_{i-1}}}{q_{j_i} - q_{j_{i-1}}}\right)\right](r_{j_k} - c_{j_k}) \\
&\quad - \left[1 - F\left(\frac{r_{j_m} - r_{j_{m-1}}}{q_{j_m} - q_{j_{m-1}}}\right)\right](r_{j_k} - c_{j_k}) \\
&= 0.
\end{aligned}$$

Hence, $S^*$ cannot be optimal. In case (2), let $l \in \{k+2, ..., m\}$ be the smallest integer $r_{j_k} - c_{j_k} \leq r_{j_l} - c_{j_l}$ and let $S = \{j_1, ..., j_k, j_l, ..., j_m\}$.

$$
\begin{aligned}
& \mathbb{E}\Pi(S) - \mathbb{E}\Pi(S^*) \\
= & \left[F\left(\frac{r_{j_l} - r_{j_k}}{q_{j_l} - q_{j_k}}\right) - F\left(\frac{r_{j_k} - r_{j_{k-1}}}{q_{j_k} - q_{j_{k-1}}}\right)\right](r_{j_k} - c_{j_k}) \\
& - \left[F\left(\frac{r_{j_{k+1}} - r_{j_k}}{q_{j_{k+1}} - q_{j_k}}\right) - F\left(\frac{r_{j_k} - r_{j_{k-1}}}{q_{j_k} - q_{j_{k-1}}}\right)\right](r_{j_k} - c_{j_k}) \\
& - \sum_{i=k+1}^{l-1}\left[F\left(\frac{r_{j_{i+1}} - r_{j_i}}{q_{j_{i+1}} - q_{j_i}}\right) - F\left(\frac{r_{j_i} - r_{j_{i-1}}}{q_{j_i} - q_{j_{i-1}}}\right)\right](r_{j_i} - c_{j_i}) \\
> & \left[F\left(\frac{r_{j_l} - r_{j_k}}{q_{j_l} - q_{j_k}}\right) - F\left(\frac{r_{j_k} - r_{j_{k-1}}}{q_{j_k} - q_{j_{k-1}}}\right)\right](r_{j_k} - c_{j_k}) \\
& - \left[F\left(\frac{r_{j_{k+1}} - r_{j_k}}{q_{j_{k+1}} - q_{j_k}}\right) - F\left(\frac{r_{j_k} - r_{j_{k-1}}}{q_{j_k} - q_{j_{k-1}}}\right)\right](r_{j_k} - c_{j_k}) \\
& - \left[F\left(\frac{r_{j_l} - r_{j_k}}{q_{j_l} - q_{j_k}}\right) - F\left(\frac{r_{j_{k+1}} - r_{j_k}}{q_{j_{k+1}} - q_{j_k}}\right)\right](r_{j_k} - c_{j_k}) \\
& - \left[F\left(\frac{r_{j_l} - r_{j_{l-1}}}{q_{j_l} - q_{j_{l-1}}}\right) - F\left(\frac{r_{j_l} - r_{j_k}}{q_{j_l} - q_{j_k}}\right)\right](r_{j_l} - c_{j_l}) \\
= & \ 0
\end{aligned}
$$

Where the inequality comes from $\frac{r_{j_k}-r_{j_{k-1}}}{q_{j_k}-q_{j_{k-1}}} < \frac{r_{j_l}-r_{j_k}}{q_{j_l}-q_{j_k}} < \frac{r_{j_{l+1}}-r_{j_l}}{q_{j_{l+1}}-q_{j_l}}$ and the fact that $(r_{j_i} - c_{j_i}) < r_{j_k} - c_{j_k} < r_{j_l} - c_{j_l}$ for $i = k+1, ..., l-1$. Hence $S^*$ cannot be optimal and we have a contradiction. □

*Proof of Theorem* 4.4.5. Let $S = \{j_1, \ldots, j_m\}$ with $j_1 < ... < j_m$ and $p(S)$ be the path that corresponds to $S$, where $p(S) = (0,0) \to (0, j_1) \to (j_1, j_2) \to ... \to (j_{m-1}, j_m) \to (j_m, n+1) \to (n+1, n+1)$. Let $P$ be the set of paths. Every set $S$ that satisfies the condition of Lemma 3.3.1 corresponds to a path in $P$ and vice versa.

The cost of path $p(S)$ is equal to:

$$\begin{aligned}
C(p(S)) &= C_{(0,0),(0,j_1)} + C_{(0,j_1),(j_1,j_2)} + \ldots + C_{(j_{m-1},j_m),(j_m,n+1)} \\
&\quad + C_{(j_m,n+1),(n+1,n+1)}, \\
&= mK - \sum_{i=1}^{m-1}(r_{j_i} - c_{j_i})\left[F\left(\frac{r_{j_{i+1}} - r_{j_i}}{q_{j_{i+1}} - q_{j_i}}\right) - F\left(\frac{r_{j_i} - r_{j_{i-1}}}{q_{j_i} - q_{j_{i-1}}}\right)\right] \\
&\quad -(r_{j_m} - c_{j_m})\left[1 - F\left(\frac{r_{j_m} - r_{j_{m-1}}}{q_{j_m} - q_{j_{m-1}}}\right)\right] \\
&= -\mathbb{E}\Pi(S).
\end{aligned}$$

Hence, $\min_{p\in P} C(p(S)) = \min_{S} \left[-\mathbb{E}\Pi(S)\right] = \max_{S} \mathbb{E}\Pi(S)$. □

*Proof of Corollary* 3.3.2. The complexity of a shortest path problem in an acyclic network is bounded by the number of arcs (see Ahuja *et al.* (1993) page 107). The graph has a special structure, because there is possibly an arc between two nodes $(i,j)$ to $(l,k)$ only if $j = l$. There are at most $j$ nodes that end with product $j \in \{1, ...n\}$ and these are connected to at most $n+1-j$ nodes that start with product $j$. Therefore the maximum number of arcs is equal to $2n + \sum_{j=1}^{n}(n+1-j)j$, where $2n$ is the maximum number of nodes leaving the source or ending in the destination node. Hence, the maximum number of arcs is $O(n^3)$. □

*Proof of Lemma* 3.4.1. We can write the expected profit function as

$$\mathbb{E}\Pi(\vec{\theta}) = \sum_{i=1}^{m}[1 - F(\theta_{j_i})][\theta_{j_i}(q_{j_i} - q_{j_{i-1}}) - (c_{j_i} - c_{j_{i-1}})] - mK.$$

Taking the derivative of the expected profit with respect to $r_{j_i}$ for $i =$

$1, ..., m$, we get

$$\frac{\partial \mathbb{E}\Pi}{\partial r_{j_i}} = f(\theta_{j_i})\left[\eta(\theta_{j_i}) - \theta_{j_i} + \frac{c_{j_i} - c_{j_{i-1}}}{q_{j_i} - q_{j_{i-1}}}\right].$$

At the first order conditions (FOC), we have

$$\theta_{j_i} = \eta(\theta_{j_i}) + \frac{c_{j_i} - c_{j_{i-1}}}{q_{j_i} - q_{j_{i-1}}} \qquad \text{for } i = 1, ..., m. \tag{B.1}$$

Also,

$$\left.\frac{\partial \mathbb{E}\Pi^2}{\partial r_{j_i} \partial r_{j_k}}\right|_{FOC} = \begin{cases} 0 & \text{for } k \notin \{i-1, i\} \\ \frac{-f(\theta_{j_i})}{q_{j_i} - q_{j_{i-1}}}[\eta'(\theta_{j_i}) - 1] > 0 & \text{for } k = i-1 \\ \frac{f(\theta_{j_i})}{q_{j_i} - q_{j_{i-1}}}[\eta'(\theta_{j_i}) - 1] < 0 & \text{for } k = i \end{cases}$$

Therefore, the Hessian matrix is negative definite and the solutions to (B.1) determine the maximum. Since $F$ is an IFR distribution, $\eta(\theta)$ is a decreasing function and therefore, each equation in (B.1) gives a unique solution. Let $\theta^*_{j_i}$, for $i = 1, ..., m$, denote the solutions to (B.1). By (4.1) and the definition of $S^*$, the solution must satisfy $\theta^*_{j_1} < \theta^*_{j_2} < ... < \theta^*_{j_m} < \bar{\theta}$, therefore we need $\frac{c_{j_1}}{q_{j_1}} < \frac{c_{j_2} - c_{j_1}}{q_{j_2} - q_{j_1}} < ... < \frac{c_{j_m} - c_{j_{m-1}}}{q_{j_m} - q_{j_{m-1}}} < \bar{\theta}$, which proves that (3.11) is a necessary condition. Finally, (3.12) and (3.13) follow directly from (3.11). $\square$

*Proof of Lemma* 3.4.2. The proof of Lemma 3.4.1 shows that the first order conditions determine the optimal solutions; therefore, the values of $\theta^*_{j_i}$ for $i = 1, ..., m$ can be obtained by solving (4.5). Moreover, we can get the prices $\vec{r}^*$ by using $\theta^*_j = \frac{r^*_j - r^*_{j-1}}{q_j - q_{j-1}}$ for $j = 1, ..., n$. $\square$

*Proof of Corollary* 3.4.3. Notice that (3.16) follows directly from (3.15).

To prove (3.17), notice that $r^*_{j_i} - c_{j_i} = \sum_{k=1}^{i}(q_{j_k} - q_{j_{k-1}})\eta(\theta^*_{j_k})$ for $i = 1, ..., m$ which is strictly increasing in $i$. $\square$

*Proof of Lemma* 3.4.4. First we show that $S^* = \{j_1, ..., j_m\}$ with $j_1 < ... < j_m$ cannot contain a dominated product along with a product that dominates it. Suppose not (contradiction), then there must exist $j_i$ and $j_{i+1}$ such that $j_{i+1}$ dominates $j_i$. In this case we would have $\frac{c_{j_{i+1}} - c_{j_i}}{q_{j_{i+1}} - q_{j_i}} < 0$ which contradicts (3.11) from Lemma 3.4.1.

Now, suppose $S^*$ contains a dominated product but the product(s) that dominate(s) it are not in $S^*$. In this case, there must exists $j_i$ for some $i = 1, ..., m$ which is dominated by $k$ where $j_i < k < j_{i+1}$. Given that $c_k < c_{j_i}$ and $q_k > q_{j_i}$,

$$\frac{c_k - c_{j_i}}{q_k - q_{j_i}} < 0. \tag{B.2}$$

Let $S = \{j_1, ..., j_{i-1}, k, j_{i+1}, ..., j_m\}$. We know from Lemma 3.4.2 that, in $S^*$, $\theta^*_{j_i}$ are obtained using (4.5). In $S$, let $\theta_{j_x} = \theta^*_{j_x}$ for $x = 1, ..., i-1, i+1, ...., m$ and $\theta_k = \theta^*_{j_i}$ for $i = 1, ..., m$. We have

$$\begin{aligned}
\mathbb{E}\Pi(S) - \mathbb{E}\Pi(S^*) &= [1 - F(\theta^*_{j_i})][\theta^*_{j_i}(q_k - q_{j_i}) - (c_k - c_{j_i})] \\
&\quad -[1 - F(\theta^*_{j_{i+1}})][\theta^*_{j_{i+1}}(q_k - q_{j_i}) - (c_k - c_{j_i})] \\
&= (q_k - q_{j_i})[1 - F(\theta^*_{j_i})]\left(\theta^*_{j_i} - \frac{c_k - c_{j_i}}{q_k - q_{j_i}}\right) \\
&\quad -(q_k - q_{j_i})[1 - F(\theta^*_{j_{i+1}})]\left(\theta^*_{j_{i+1}} - \frac{c_k - c_{j_i}}{q_k - q_{j_i}}\right)
\end{aligned}$$

Let $\psi(\theta) = [1 - F(\theta)]\left[\theta - \frac{c_k - c_{j_i}}{q_k - q_{j_i}}\right]$. We have $\psi'(\theta) = f(\theta)\left[\eta(\theta) + \frac{c_k - c_{j_i}}{q_k - q_{j_i}} - \theta\right] <$

0 due to condition (B.2) and the FOC (B.1). Therefore, $\mathbb{E}\Pi(S) - \mathbb{E}\Pi(S^*) > 0$ due to the fact that $\theta^*_{j_{i+1}} > \theta^*_{j_i}$ and $q_k > q_{j_i}$. Hence, $S^*$ is not optimal, which is a contradiction. $\square$

*Proof of Theorem* 3.4.1. The proof is very similar to that of Theorem 4.4.5 and is therefore omitted. $\square$

*Proof of Corollary* 3.4.5. The proof is very similar to that of Corollary 3.3.2 and is therefore omitted. $\square$

*Proof of Lemma* 3.4.6. Let $j_0 = 0$. We first show that (4.4)-(3.19) are necessary conditions. We prove (4.4) by contradiction, that is, suppose that $S^*$ is optimal but there exists $k \in \{j_i + 1, ..., j_{i+1} - 1\}$ such that $\frac{c_k - c_{j_i}}{q_k - q_{j_i}} < \frac{c_{j_{i+1}} - c_k}{q_{j_{i+1}} - q_k}$.

Using the fact that $\frac{a}{b} < \frac{c}{d}$ implies that $\frac{a}{b} < \frac{a+c}{b+d}$ when $a, b, c, d > 0$, this implies that

$$\frac{c_k - c_{j_i}}{q_k - q_{j_i}} < \frac{c_{j_{i+1}} - c_{j_i}}{q_{j_{i+1}} - q_{j_i}}. \tag{B.3}$$

Let $S = \{j_1, ..., j_i, k, j_{i+1}, ..., j_m\}$. We know from Lemma 3.4.2 that, in $S^*$, $\theta^*_{j_i}$ are obtained using (4.5). In $S$, let $\theta_{j_i} = \theta^*_{j_i}$ for $i = 1, ..., m$. If

$$\frac{c_k - c_{j_i}}{q_k - q_{j_i}} > \frac{c_{j_i} - c_{j_{i-1}}}{q_{j_i} - q_{j_{i-1}}}, \tag{B.4}$$

let $\theta_k$ be the solution to $\theta_k = \eta(\theta_k) + \frac{c_k - c_{j_i}}{q_k - q_{j_i}}$; otherwise, let $\theta_k$ be a value such that

$\theta^*_{j_i} < \theta_k < \theta^*_{j_{i+1}}$. We have

$$\begin{aligned}
\mathbb{E}\Pi(S) - \mathbb{E}\Pi(S^*) &= [1 - F(\theta_k)][\theta_k(q_k - q_{j_i}) - (c_k - c_{j_i})] \\
&\quad -[1 - F(\theta^*_{j_{i+1}})][\theta^*_{j_{i+1}}(q_k - q_{j_i}) - (c_k - c_{j_i})] \\
&= (q_k - q_{j_i})[1 - F(\theta_k)] \left(\theta_k - \frac{c_k - c_{j_i}}{q_k - q_{j_i}}\right) \\
&\quad -(q_k - q_{j_i})[1 - F(\theta^*_{j_{i+1}})] \left(\theta^*_{j_{i+1}} - \frac{c_k - c_{j_i}}{q_k - q_{j_i}}\right)
\end{aligned}$$

Let $\psi(\theta) = [1-F(\theta)]\left[\theta - \frac{c_k - c_{j_i}}{q_k - q_{j_i}}\right]$. We have $\psi'(\theta) = f(\theta)\left[\eta(\theta) + \frac{c_k - c_{j_i}}{q_k - q_{j_i}} - \theta\right]$. If condition (B.4) is satisfied, then $\psi'(\theta_k) = 0$ and $\psi'(\theta) < 0$ for $\theta > \theta_k$; otherwise, $\psi'(\theta) \leq 0$ for $\theta \geq \theta_k$, so $\psi(\theta)$ is decreasing in $\theta$ for $\theta \geq \theta_k$. Therefore $\mathbb{E}\Pi(S) - \mathbb{E}\Pi(S^*) > 0$ due to the fact that $\theta^*_{j_{i+1}} > \theta_k$ and $q_k > q_{j_i}$. Hence, $S^*$ is not optimal, which is a contradiction.

Finally, we prove that (3.19) is a necessary condition. Suppose (contradiction) that $S^*$ is optimal with $j_m < n$ and there exists $k \in \{j_{m+1}, ..., n\}$ such that $\frac{c_k - c_{j_m}}{q_k - q_{j_m}} < \bar{\theta}$. Let $S = \{j_1, ..., j_m, k\}$. In $S$, let $\theta_{j_i} = \theta^*_{j_i}$ for $i = 1, ..., m$ and let $\theta_k$ be any value such that $\theta_k < \frac{c_k - c_{j_m}}{q_k - q_{j_m}} < \overline{\theta}$. We have

$$\begin{aligned}
\mathbb{E}\Pi(S) - \mathbb{E}\Pi(S^*) &= [1 - F(\theta_k)]\left[\theta_k(q_k - q_{j_m}) - (c_k - c_{j_m})\right] \\
&= [1 - F(\theta_k)](q_k - q_{j_m})\left(\theta_k - \frac{c_k - c_{j_m}}{q_k - q_{j_m}}\right) > 0.
\end{aligned}$$

Therefore $S^*$ cannot be optimal and we have a contradiction.

Now we prove that (3.11), (4.4) and (3.19) are sufficient conditions by showing only one set $S^*$ satisfies these conditions. Suppose we have two sets

$S_1^* = \{j_1^1, ..., j_{m_1}^1\}$ such that $j_1^1 < ... < j_{m_1}^1$ and $S_2^* = \{j_1^2, ..., j_{m_2}^2\}$ such that $j_1^2 < ... < j_{m_2}^2$ satisfying (3.11), (4.4) and (3.19).

If we have $j_k^1 = j_k^2$ for $k = 1, ..., m_1$ and $m_2 > m_1$, then (3.19) applied to $S_1^*$ implies that $\frac{c_{j_{m_1+1}^2} - c_{j_{m_1}^2}}{q_{j_{m_1+1}^2} - q_{j_{m_1}^2}} \geq \bar{\theta}$. However this contradicts (3.11) for $S_2^*$. Therefore, we exclude this case. Without loss of generality, let $i$ be the smallest integer such that $j_k^1 = j_k^2$ for $k = 1, ..., i-1$ and $j_i^1 < j_i^2$. We have two cases: (1) $j_{m_1}^1 \geq j_i^2$, in this case let $l$ be the smallest integer such that $j_l^1 \geq j_i^2$; (2) $j_{m_1}^1 < j_i^2$.

Let us consider Case (1) first. Since $S_1^*$ satisfies (3.11), we have $\frac{c_{j_1^1}}{q_{j_1^1}} < ... < \frac{c_{j_l^1} - c_{j_{l-1}^1}}{q_{j_l^1} - q_{j_{l-1}^1}}$, which implies,

$$\frac{c_{j_{l-1}^1} - c_{j_{i-1}^1}}{q_{j_{l-1}^1} - q_{j_{i-1}^1}} < \frac{c_{j_l^1} - c_{j_{l-1}^1}}{q_{j_l^1} - q_{j_{l-1}^1}} \text{ for } i = 1, ..., l-2 \tag{B.5}$$

Due to $j_{i-1}^1 = j_{i-1}^2$, if $j_l^1 = j_i^2$, this contradicts with (4.4) for set $S_2^*$ as (B.5) is equivalent to $\frac{c_k - c_{j_{i-1}^2}}{q_k - q_{j_{i-1}^2}} < \frac{c_{j_i^2} - c_k}{q_{j_i^2} - q_k}$, with $k = j_{l-1}^1$. So let us assume that $j_l^1 > j_i^2$. Since $S_1^*$ satisfies (4.4) and $j_{l-1}^1 < j_i^2$, $\frac{c_{j_i^2} - c_{j_{l-1}^1}}{q_{j_i^2} - q_{j_{l-1}^1}} > \frac{c_{j_l^1} - c_{j_i^2}}{q_{j_l^1} - q_{j_i^2}}$, which implies

$$\frac{c_{j_i^2} - c_{j_{l-1}^1}}{q_{j_i^2} - q_{j_{l-1}^1}} > \frac{c_{j_l^1} - c_{j_{l-1}^1}}{q_{j_l^1} - q_{j_{l-1}^1}}. \tag{B.6}$$

From (B.5) and (B.6), we get $\frac{c_{j_i^2} - c_{j_{l-1}^1}}{q_{j_i^2} - q_{j_{l-1}^1}} > \frac{c_{j_{l-1}^1} - c_{j_{i-1}^1}}{q_{j_{l-1}^1} - q_{j_{i-1}^1}}$, which contradicts (4.4) for set $S_2^*$ because it is equivalent to $\frac{c_{j_i^2} - c_k}{q_{j_i^2} - q_k} > \frac{c_k - c_{j_{i-1}^2}}{q_k - q_{j_{i-1}^2}}$ with $k = j_{l-1}^1$.

Let us now consider Case (2), i.e., $j_{m_1}^1 < j_i^2$. From (3.19) of set $S_1^*$, we get $\frac{c_{j_i^2} - c_{j_{m_1}^1}}{q_{j_i^2} - q_{j_{m_1}^1}} > \bar{\theta}$.

From (4.4) for set $S_2^*$ we have $\frac{c_{j^1_{m_1}} - c_{j^2_{i-1}}}{q_{j^1_{m_1}} - q_{j^2_{i-1}}} \geq \frac{c_{j^2_i} - c_{j^1_{m_1}}}{q_{j^2_i} - q_{j^1_{m_1}}}$.. This implies that $\frac{c_{j^1_{m_1}} - c_{j^1_{i-1}}}{q_{j^1_{m_1}} - q_{j^1_{i-1}}} > \bar{\theta}$ However, by (3.11) for $S_1^*$, we must have $\frac{c_{j^1_{m_1}} - c_{j^1_{i-1}}}{q_{j^1_{m_1}} - q_{j^1_{i-1}}} < \frac{c_{j^1_{m_1}} - c_{j^1_{m_1-1}}}{q_{j^1_{m_1}} - q_{j^1_{m_1-1}}} < \bar{\theta}$, which is a contradiction.

Therefore, there exists only one set satisfying (3.11), (4.4) and (3.19), and thus, if a set $S^*$ satisfies these conditions, then this set must be the only optimal set. $\square$

*Proof of corollary* 3.4.7. Follows directly from Lemma 3.4.6. $\square$

*Proof of Proposition* 3.4.8. Let $S^* = \{j_1, ..., j_m\}$ with $j_1 < ... < j_m$. To prove the optimality of $S^*$ we show that the conditions of Lemma 3.4.6 are satisfied. Directly from the second inequality in Step 1 in the algorithm, we obtain that (4.4) of Lemma 3.4.6 is satisfied.

We now prove by induction that (3.19) of Lemma 3.4.6 also holds. When the algorithm stops, $i = j_m$. If $j_m = n$, then (3.19) is trivially true. If $j_m < n$, it must be that $\frac{c_{j_m+1} - c_{j_m}}{q_{j_m+1} - q_{j_m}} \geq \bar{\theta}$, since otherwise the algorithm would have added $j_m + 1$ to set $S^*$. Now let us assume that $\frac{c_y - c_{j_m}}{q_y - q_{j_m}} \geq \bar{\theta}$ for $y \in \{j_m + 1, ..., x\}$ and $x \in \{j_m + 1, ..., n - 1\}$ and prove that $\frac{c_{x+1} - c_{j_m}}{q_{x+1} - q_{j_m}} \geq \bar{\theta}$.

Because $x + 1$ was not added to set $S^*$ when $i = j_m$, it must be that at least one of the following condition is true:

$$\frac{c_{x+1} - c_{j_m}}{q_{x+1} - q_{j_m}} \geq \bar{\theta} \quad \text{or} \quad \frac{c_k - c_{j_m}}{q_k - q_{j_m}} \leq \frac{c_{x+1} - c_k}{q_{x+1} - q_k} \text{ for some } k \in \{j_m + 1, ..., x\}.$$

If the first condition is true, then we are done. If the second condition is true, then

$$\frac{c_k - c_{j_m}}{q_k - q_{j_m}} \leq \frac{c_{x+1} - c_{j_m}}{q_{x+1} - q_{j_m}},$$

By the induction hypothesis, we know that $\frac{c_k - c_{j_m}}{q_k - q_{j_m}} \geq \bar{\theta}$, therefore $\frac{c_{x+1} - c_{j_m}}{q_{x+1} - q_{j_m}} \geq \bar{\theta}$. Hence, (3.19) of Lemma 3.4.6 also holds.

We now prove that (3.11) of Lemma 3.4.1 holds. Since, the algorithm adds $j_m$, it must be that $\frac{c_{j_m} - c_{j_{m-1}}}{q_{j_m} - q_{j_{m-1}}} < \bar{\theta}$. Now assume (contradiction) that

$$\frac{c_{j_x} - c_{j_{x-1}}}{q_{j_x} - q_{j_{x-1}}} \geq \frac{c_{j_{x+1}} - c_{j_x}}{q_{j_{x+1}} - q_{j_x}} \text{ for some } x \in \{1, ..., m-1\}. \tag{B.7}$$

This implies that $\frac{c_{j_x} - c_{j_{x-1}}}{q_{j_x} - q_{j_{x-1}}} \geq \frac{c_{j_{x+1}} - c_{j_{x-1}}}{q_{j_{x+1}} - q_{j_{x-1}}}$. Since $j_x$ was added by the algorithm, it must be that $\frac{c_{j_x} - c_{j_{x-1}}}{q_{j_x} - q_{j_{x-1}}} < \bar{\theta}$. Therefore,

$$\frac{c_{j_{x+1}} - c_{j_{x-1}}}{q_{j_{x+1}} - q_{j_{x-1}}} < \bar{\theta}. \tag{B.8}$$

The fact that the algorithm added $j_x$ to set $S^*$ also implies that the following conditions must have been satisfied:

$$\frac{c_k - c_{j_{x-1}}}{q_k - q_{j_{x-1}}} > \frac{c_{j_x} - c_k}{q_{j_x} - q_k} \text{ for } k = j_{x-1}+1, ..., j_x - 1 \tag{B.9}$$

which implies $\frac{c_k - c_{j_{x-1}}}{q_k - q_{j_{x-1}}} > \frac{c_{j_x} - c_{j_{x-1}}}{q_{j_x} - q_{j_{x-1}}}$ for $k = j_{x-1}+1, ..., j_x - 1$ and therefore using (B.7) we have

$$\frac{c_k - c_{j_{x-1}}}{q_k - q_{j_{x-1}}} > \frac{c_{j_{x+1}} - c_{j_x}}{q_{j_{x+1}} - q_{j_x}} \text{ for } k = j_{x-1}+1, ..., j_x - 1 \tag{B.10}$$

Combining (B.9) and (B.10) and using the fact that $\frac{a}{b} > \frac{c}{d}$ and $\frac{a}{b} > \frac{e}{f}$ implies that $\frac{a}{b} > \frac{c+e}{d+f}$ when $a, b, c, d, e, f > 0$, we obtain

$$\frac{c_k - c_{j_{x-1}}}{q_k - q_{j_{x-1}}} > \frac{c_{j_{x+1}} - c_k}{q_{j_{x+1}} - q_k} \text{ for } k = j_{x-1} + 1, ..., j_x - 1 \tag{B.11}$$

Similarly, the fact that the algorithm added $j_{x+1}$ to set $S^*$ implies that the following conditions must have been satisfied:

$$\frac{c_k - c_{j_x}}{q_k - q_{j_x}} > \frac{c_{j_{x+1}} - c_k}{q_{j_{x+1}} - q_k} \text{ for } k = j_x + 1, ..., j_{x+1} - 1 \tag{B.12}$$

which implies $\frac{c_{j_{x+1}} - c_{j_x}}{q_{j_{x+1}} - q_{j_x}} > \frac{c_{j_{x+1}} - c_k}{q_{j_{x+1}} - q_k}$ for $k = j_x + 1, ..., j_{x+1} - 1$ and therefore using (B.7) we have

$$\frac{c_{j_x} - c_{j_{x-1}}}{q_{j_x} - q_{j_{x-1}}} > \frac{c_{j_{x+1}} - c_k}{q_{j_{x+1}} - q_k} \text{ for } k = j_x + 1, ..., j_{x+1} - 1 \tag{B.13}$$

Combining (B.12) and (B.13), we obtain

$$\frac{c_k - c_{j_{x-1}}}{q_k - q_{j_{x-1}}} > \frac{c_{j_{x+1}} - c_k}{q_{j_{x+1}} - q_k} \text{ for } k = j_x + 1, ..., j_{x+1} - 1 \tag{B.14}$$

However, if (B.7), (B.8), (B.11) and (B.14) were true, then the algorithm would have added $j_{x+1}$ instead of $j_x$ when $i = j_{x-1}$. Therefore we have a contradiction.

Since (3.12) and (3.13) of Lemma 3.4.1 follow from (3.11), we have proven that all five conditions of Lemma 3.4.6 are satisfied and therefore $S^*$ is optimal.

$\square$

*Proof of Corollary* 3.4.9. In the worst-case scenario, Step 1 has to be repeated for $i$ going from 0 to $n$. In each iteration we may have to consider up to $n - i$ values of $j$. $\square$

*Proof of Corollary* 3.4.10. First we show that for every $k$ which satisfies (4.4), there cannot be at least two $\theta$ values such that $\theta q_k - c_k = \max_{i=1,...,n}(\theta q_i - c_i)$. Since $q_k > q_{j_i}$, we have $\theta q_k - c_k < \theta q_{j_i} - c_{j_i}$ for $\theta < \frac{c_k - c_{j_i}}{q_k - q_{j_i}}$. Since $q_k < q_{j_{i+1}}$, we have $\theta q_k - c_k < \theta q_{j_{i+1}} - c_{j_{i+1}}$ for $\theta > \frac{c_{j_{i+1}} - c_k}{q_{j_{i+1}} - q_k}$. We get the result since, by (4.4), $\frac{c_k - c_{j_i}}{q_k - q_{j_i}} \geq \frac{c_{j_{i+1}} - c_k}{q_{j_{i+1}} - q_k}$.

Next we show that for every $k$ that satisfies (3.19), there cannot be at least two values of $\theta < \overline{\theta}$ such that $\theta q_k - c_k = \max_{i=1,...,n}(\theta q_i - c_i)$. Since $q_k > q_{j_m}$, we have $\theta q_k - c_k < \theta q_{j_m} - c_{j_m}$ for $\theta < \frac{c_k - c_{j_m}}{q_k - q_{j_m}}$. We get the result since by (3.19), $\frac{c_k - c_{j_m}}{q_k - q_{j_m}} \geq \overline{\theta}$. Finally we show that for every $j_i, i = 1, ..., m$, there exists at least two values of $\theta \in [\underline{\theta}, \overline{\theta}]$ such that $\theta q_{j_i} - c_{j_i} = \max_{k=1,...,n}(\theta q_k - c_k)$. From (3.11), $\frac{c_{j_i} - c_{j_{i-1}}}{q_{j_i} - q_{j_{i-1}}} < \frac{c_{j_{i+1}} - c_{j_i}}{q_{j_{i+1}} - q_{j_i}}$ and therefore, $\theta q_{j_i} - c_{j_i} = \max_{k=1,...,n}(\theta q_k - c_k)$ for $\theta \in \left(\frac{c_{j_i} - c_{j_{i-1}}}{q_{j_i} - q_{j_{i-1}}}, \frac{c_{j_{i+1}} - c_{j_i}}{q_{j_{i+1}} - q_{j_i}}\right)$. □

*Proof of Lemma* 3.4.11. From (4.5), we get

$$\theta^*_{j_i} = \frac{\beta + \frac{c_{j_i} - c_{j_{i-1}}}{q_{j_i} - q_{j_{i-1}}}}{1 - \alpha}. \tag{B.15}$$

Therefore, substituting (B.15) to (3.15) gives (3.20), which implies that $r^*_{j_i}$ only depends on $c_{j_i}$ and $q_{j_i}$. □

*Proof of proposition* 3.4.12. Lemma 3.4.11 indicates that the prices of products in the optimal assortment are determined by (3.20), i.e., such that each price is a function of its own cost and quality level only. Therefore, we can solve the assortment planning problem with exogenous prices defined by (3.20). □

*Proof of Proposition* 3.4.13. From Proposition 3.4.12, we know that the products that are offered are priced using (3.20). Let $S = \{j_1, ..., j_m\}$ with $j_1 < ... < j_m$ be the set of products such that $P_{j_i}(\{1, ..., n\}) > 0$ when $r_{j_i} = \frac{c_{j_i} + bq_{j_i}}{1-a}$. In other words, we have $P_k(\{1, ..., n\}) = 0$ for $k \notin S$. We show that $S$ satisfies the conditions of Lemmas 3.4.1 and 3.4.6.

First, from the definition of $S$ it must be that $\frac{r_{j_1}}{q_{j_1}} < \frac{r_{j_2} - r_{j_1}}{q_{j_2} - q_{j_1}} < ... < \frac{r_{j_m} - r_{j_{m-1}}}{q_{j_m} - q_{j_{m-1}}} < \bar{\theta}$. Using condition (3.20), we get $\frac{c_{j_1}}{q_{j_1}} < \frac{c_{j_2} - c_{j_1}}{q_{j_2} - q_{j_1}} < ... < \frac{c_{j_m} - c_{j_{m-1}}}{q_{j_m} - q_{j_{m-1}}} < \bar{\theta}$, which is (3.11) from Lemma 3.4.1. The other conditions from Lemma 3.4.1 follow from this one.

For $k$ such that $j_i < k < j_{i+1}$, By definition of $S$, it must be that $\frac{r_k - r_{j_i}}{q_k - q_{j_i}} > \frac{r_{j_{i+1}} - r_k}{q_{j_{i+1}} - q_k}$ Using condition (3.20), we get $\frac{c_k - c_{j_i}}{q_k - q_j} > \frac{c_{j_{i+1}} - c_k}{q_{j_{i+1}} - q_k}$, which is (4.4) from Lemma 3.4.6.

Now consider $k$ such that $k > j_m$. By definition of $S$, it must be that $\frac{r_k - r_{j_m}}{q_k - q_{j_m}} \geq \bar{\theta}$. Using condition (3.20), we get $\frac{c_k - c_{j_m}}{q_k - q_{j_m}} \geq \bar{\theta}$, which is condition (3.19) from Lemma 3.4.6.

It follows that $S$ satisfies all the conditions of Lemma 3.4.6 and therefore it is optimal. $\square$

*Proof of Proposition* 3.5.1. By Corollary 3.4.10, $S^*_{\mathcal{N}}$ is such that for $j \in S^*_{\mathcal{N}}$, there exists at least two values of $\theta \in [\underline{\theta}, \overline{\theta}]$ such that $\theta q_j - c_j = \max_{i \in \mathcal{N}}(\theta q_i - c_i)$. Consider $j \in S^*_{\mathcal{N}} \cap \mathcal{N}'$, it must be true that there exists at least two values of $\theta \in [\underline{\theta}, \overline{\theta}]$ such that $\theta q_j - c_j = \max_{i \in \mathcal{N}'}(\theta q_i - c_i)$. Therefore, $j \in S^*_{\mathcal{N}'}$. $\square$

# Appendix C

# Supplemental Material for Chapter 4

*Proof of Lemma 4.4.1.* The result follows from Lemmas 3.4.1 and 3.4.6 $\square$

*Proof of Lemma 4.4.2.* The result follows from Lemmas 3.4.1 and 3.4.6. $\square$

*Proof of Lemma 4.4.3.* Barghava & Choudhary (2001) show that (4.6) is a sufficient condition. Next we show that (4.6) is a necessary condition, i.e., if the optimal set $S^*$ includes the bundle, then condition (4.6) must be satisfied.

Suppose $S^* = \{j_1, ..., j_m\}$ such that $j_1 < j_2 < ... < j_m$ and $j_m$ is the bundle, i.e., $j_m = n + 1$. We first prove that

$$\frac{c_{\{j_{m-1}+1\}} + ... + c_{j_m}}{q_{\{j_{m-1}+1\}} + ... + q_{j_m}} < \overline{\theta} \tag{C.1}$$

Suppose (C.1) is not satisfied. Then, by (4.5),

$$\theta_m^* = \eta(\theta_m^*) + \frac{c_{\{j_{m-1}+1\}} + ... + c_{j_m}}{q_{\{j_{m-1}+1\}} + ... + q_{j_m}} > \overline{\theta}$$

which contradicts the fact that $j_m$ is included in $S^*$. Hence, (C.1) holds.

Using the fact that $\frac{a}{b} < \frac{c}{d}$ implies that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ for $a, b, c, d > 0$ along with (4.3), we obtain that

$$\frac{c_{\{j_i+1\}} + ... + c_{j_{i+1}}}{q_{\{j_i+1\}} + ... + q_{j_{i+1}}} < \frac{c_{\{j_i+1\}} + ... + c_{j_m}}{q_{\{j_i+1\}} + ... + q_{j_m}} < \frac{c_{\{j_{m-1}+1\}} + ... + c_{j_m}}{q_{\{j_{m-1}+1\}} + ... + q_{j_m}} \quad \text{for } i = 0, ..., m-2 \tag{C.2}$$

where $c_{j_0} = c_0 = 0$. Therefore, using (C.1) we get,

$$\frac{c_{\{j_i+1\}} + ... + c_{j_m}}{q_{\{j_i+1\}} + ... + q_{j_m}} < \bar{\theta} \qquad \text{for } i = 0, ..., m-2 \tag{C.3}$$

From (4.4), we get for $i = 0, ..., m-1$

$$\frac{c_{k+1} + ... + c_{j_{i+1}}}{q_{k+1} + ... + q_{j_{i+1}}} < \frac{c_{j_i+1} + ... + c_{j_{i+1}}}{q_{j_i+1} + ... + q_{j_{i+1}}} \qquad \text{for } k = j_i + 1, ..., j_{i+1}$$

And using (C.2), we obtain

$$\frac{c_{k+1} + ... + c_{j_i}}{q_{k+1} + ... + q_{j_i}} < \frac{c_{\{j_i+1\}} + ... + c_{j_m}}{q_{\{j_i+1\}} + ... + q_{j_m}} \qquad \text{for } k = j_i + 1, ..., j_{i+1}$$

And therefore, for $i = 0, ..., m-1$,

$$\frac{c_{k+1} + ... + c_{j_m}}{q_{k+1} + ... + q_{j_m}} < \frac{c_{\{j_i+1\}} + ... + c_{j_m}}{q_{\{j_i+1\}} + ... + q_{j_m}} < \bar{\theta} \qquad \text{for } k = j_i + 1, ..., j_{i+1} \tag{C.4}$$

We obtain the result by combining (C.3) and (C.4). □

*Proof of Proposition 4.4.4.* This result follows directly from Theorem 1 from Barghava & Choudhary (2001). □

*Proof of Theorem 4.4.5.* Let $S = \{j_1, \ldots, j_m\}$ with $j_1 < ... < j_m$. Let $p(S)$ be the path that corresponds to $S$, where $p(S) = (0,0) \rightarrow (0, j_1) \rightarrow (j_1, j_2) \rightarrow$

$... \rightarrow (j_{m-1}, j_m) \rightarrow (j_m, n+2) \rightarrow (n+2, n+2)$. Let $P$ be the set of all admissible paths, *i.e.*, the paths with finite costs. Every set $S$ that satisfies the condition of Lemma 4.4.1 corresponds to a path in $P$ and vice versa.

The cost of path $p(S)$ is equal to:

$$\begin{aligned}
C(p(S)) &= c_{(0,0),(0,j_1)} + c_{(0,j_1),(j_1,j_2)} + \ldots + c_{(j_{m-1},j_m),(j_m,n+2)} + c_{(j_m,n+2),(n+2,n+2)}, \\
&= -\sum_{i=1}^{m} \left[ r_{j_{i-1}+1} + .... + r_{j_i} - (c_{j_{i-1}} + ... + c_{j_i}) \right] \left[ 1 - F(\theta_{j_i}) \right] + 0 + 0 \\
&= -\mathbb{E}\Pi.
\end{aligned}$$

Hence, $\min_{p \in P} C(p(S)) = \min_{S} (-\mathbb{E}\Pi) = \max_{S} \mathbb{E}\Pi$. □

*Proof of Corollary 4.4.6.* The complexity of a shortest path problem in an acyclic network is bounded by the number of arcs (see Ahuja *et al.* (1993) page 107). Due to the special structure, the number of arcs is $O(n^3)$. □

*Proof of proposition 4.4.7.* The proof is similar to the proof of Proposition 3.4.8 of Chapter 3 , and is therefore omitted.

*Proof of Proposition 4.4.8.* The set $\{1, 2, ..., n\}$ satisfies conditions (4.3) and (4.4), therefore, it must be the optimal set according to Lemma 4.4.1.

In the presence of the bundle, by Lemma 4.4.3, condition (4.6) implies that it is optimal to stock the bundle. Suppose the optimal set in the presence of the bundle does not include a component $k$, where $1 \leq k < j$, i.e., $S^{*'} = \{1, ..., k-1, k+1, ..., j, n+1\}$. Condition (4.4) of Lemma 4.4.1 implies that we must have $\frac{c_k}{q_k} > \frac{c_{k+1}}{q_{k+1}}$, which contradicts condition (4.8). Therefore we must have $S^{*'} = \{1, ..., j, n+1\}$. Moreover, Lemma 4.4.1 implies that product $j$ is not included in the optimal assortment if the following conditions is satisfied.

$$\frac{c_{j+1}}{q_{j+1}} > \frac{c_{j+2} + ... + c_n}{q_{j+2} + ... + q_{n+1}}$$

.

Finally, let $\theta_1, ..., \theta_j$ and $\theta'_1, ..., \theta'_j$ denote the optimal $\theta$ value in the absence of the bundle and in the presence of bundle respectively, let $r^*_1, ..., r^*_j$ and $r^{*'}_1, ..., r^{*'}_j$ denote the optimal price in the absence of the bundle and in the presence of bundle respectively. Lemma 4.4.2 implies that

$$\begin{aligned} \theta_i &= \theta'_i \quad \text{for } i = 1, ..., j \\ r_i &= r'_i \quad \text{for } i = 1, ..., j \end{aligned}$$

. $\square$

*Proof of Proposition 4.5.1* . By Lemma 4.4.3, (4.10) is the necessary and sufficient condition for the bundle to be included in the optimal solution. We first prove the results when condition (4.10) is satisfied. we suppose $S^* = \{1, 2, ..i, i+l, j, n+1\}$, where $l > 1$. For $i < k < i+l$, we have

$$\begin{aligned} &\frac{1}{q_i} < ... < \frac{1}{q_{i+k}} < ... < \frac{1}{q_{i+l}} \\ &\Rightarrow \frac{k}{q_{i+1} + ... + q_{i+k}} < \frac{l-k}{q_{i+k+1} + ... + q_{i+l}} \end{aligned}$$

which contradicts condition 4.4 of Lemma 4.4.1. Therefore, the optimal set includes the first $j$ components, i.e. $S^* = \{1, ..., j, n+1\}$. Moreover, $j$ must satisfy condition (4.11). If not, then

$$\frac{c}{q_j} > \frac{(n-j)c}{q_{j+1} + ... + q_{n+1}}$$

and the set cannot be optimal from Lemma 4.4.1.

Next we prove that $\hat{j}$ is the largest integer satisfies condition (4.11). Suppose $\hat{j}$ is not the largest integer such that condition (4.11) is satisfied. There exists $k > \hat{j}$ such that

$$\frac{1}{q_k} < \frac{(n-k)}{q_{k+1} + ... + q_{n+1}} \tag{C.5}$$

. Since we have

$$\begin{aligned}&\frac{1}{q_{j+1}} < ... < \frac{1}{q_k}\\ &\Rightarrow \frac{k-j}{q_{j+1} + ... + q_k} < \frac{1}{q_k}\end{aligned}$$

, combining with condition (C.5), we have

$$\frac{k-j}{q_{j+1} + ... + q_k} < \frac{(n-k)}{q_{k+1} + ... + q_{n+1}}$$

which contradicts with condition (4.4) of Lemma 4.4.1, for all $\hat{j} < k < n+1$,

$$\frac{k-j}{q_{j+1} + ... + q_k} > \frac{n-k}{q_{k+1} + ... + q_{n+1}}$$

. Hence, $\hat{j}$ is the largest integer satisfying condition (4.11).

If no positive integer satisfying condition (4.11), then from condition (4.3) of Lemma 4.4.1, the optimal set does not include any individual component. Intuitively, if the average component quality in the bundle is the highest, then the firm should not offer individual components.

Now we prove the results when condition (4.10) is not satisfied. By Lemma 4.4.3, we know the optimal set does not include the bundle. By the same reasoning as shown above, we know the optimal set $S^* = \{1, ..., \tilde{j}\}$ where $\tilde{j}$ is the largest integer $j$ such that $\frac{c_j}{q_j} < \bar{\theta}$. $\square$

*Proof of Proposition* (4.5.2). when condition 4.13 is satisfied, then the expected profit always jumps down at $t_j$ for $j = 1, ..., n-1$, moreover,

$$\begin{aligned} \frac{\partial \mathbb{E}\Pi}{\partial R_{n+1}} &= f(\theta_{n+1}) \left[ \eta(\theta_{n+1}) - \theta_{n+1} + \frac{(n-j)c}{Q_{n+1} - Q_j} \right] \qquad \text{for } j = 0, ..., n-1 \\ \frac{\partial \mathbb{E}^2\Pi}{\partial R_{n+1}^2} &= f(\theta_{n+1}) \left[ \eta'(\theta_{n+1}) - 1 \right] < 0 \qquad \text{for } j = 0, ..., n-1 \end{aligned}$$

so all the points satisfying the FOC conditions are local maximum points and we compare the expected profit at those points to get the global optimal point. $\square$

*Proof of Proposition 4.5.3.* Set $c = 0$, inequality (4.7) always holds and the result follows from proposition 4.4.4. $\square$

*Proof of Proposition 4.5.4.*

$$\begin{aligned} \frac{\partial \mathbb{E}\Pi}{\partial R_{n+1}} &= 1 - F(\theta_{n+1}) - f(\theta_{n+1})\theta_{n+1} \\ &= \left[1 - F(\theta_{n+1})\right] \left[1 - \xi(\theta_{n+1})\right] \end{aligned}$$

In this case, $\xi(\theta_{n+1})$ is non-decreasing in $\theta$, and $\frac{\partial^2 \mathbb{E}\Pi}{\partial R_{n+1}^2} \leq 0$ so that $\mathbb{E}\Pi$ is concave in $R_{n+1}$ and $\theta^*_{n+1}$ is the unique solution to the FOC. $\square$

*Proof of Proposition 4.5.5.* Note that both $\mathbb{E}\Pi$ and $R_{n+1}$ are continuous in $\theta_j$: at $\theta_j = \theta^*_{n+1}$, we have:

$$\begin{aligned} R^*_{n+1}\left(\theta_j^-\right) &= jr + (Q_{n+1} - Q_j)\theta^*_{n+1} \\ &= (j-1)r + (Q_{n+1} - Q_{j-1})\theta^*_{n+1} \\ &= R^*_{n+1}\left(\theta_j^+\right) \end{aligned}$$

$$\begin{aligned} \mathbb{E}\Pi^*\left(\theta_j^-\right) &= \left(1 - F(\theta^*_{n+1})\right)(Q_{n+1} - Q_j)\theta^*_{n+1} + \sum_{i=1}^{j}(1 - F(\theta_i))r \\ &= (1 - F(\theta^*_{n+1}))(Q_{n+1} - Q_{j-1})\theta^*_{n+1} + \sum_{i=1}^{j-1}(1 - F(\theta_i))r \\ &= \mathbb{E}\Pi^*\left(\theta_j^+\right) \end{aligned}$$

Hence, it is enough to consider only the first derivative when looking at the impact of changes in $q_{n+1}, q_j$, and $r$.

Regarding $q_{n+1}$, we have $\frac{\partial R^*_{n+1}}{\partial q_{n+1}} = \theta^*_{n+1}$ and $\frac{\partial \mathbb{E}\Pi(R^*_{n+1})}{\partial q_{n+1}} = (1-F(\theta^*_{n+1}))\theta^*_{n+1} > 0$. So, $\mathbb{E}\Pi^*$ and $R^*_{n+1}$ are both increasing in $q_{n+1}$.

We vary $q_j$ for some $j$ while keeping $q_1, ..., q_{j-1}, q_{j+1}, ...q_n$ constant. Because the products are always numbered such that $q_1 \geq q_2 \geq ... \geq q_n$, it is enough to consider varying $q_j \in [q_{j+1}, q_{j-1})$ for $j = 1, ..., n$ with $q_0 = \infty$.

There are two cases:

- Case 1: $j$ is such that $\theta_j \leq \theta^*_{n+1}$ (which implies that $j \leq j^*$), then $\frac{\partial R^*_{n+1}}{\partial q_j} = 0$

and $\frac{\partial \mathbb{E}\Pi^*}{\partial q_j} = f(\theta_j)\theta_j^2 > 0$.

- Case 2: $j$ is such that $\theta_j > \theta_{n+1}^*$ (which implies $j > j^*$), then $\frac{\partial R_{n+1}^*}{\partial q_j} = \theta_{n+1}^* > 0$ and $\frac{\partial \mathbb{E}\Pi^*}{\partial q_j} = (1 - F(\theta_{n+1}^*))\theta_{n+1}^* > 0$.

Hence, $\mathbb{E}\Pi^*$ is increasing in $q_j$ and $r_b^*$ is non-decreasing in $q_j$.

Regarding $r$, we have $\frac{\partial R_{n+1}^*}{\partial r} = j^* \geq 0$ and $\frac{\partial \mathbb{E}\Pi(R_{n+1}^*)}{\partial r} = \sum_{i=1}^{j^*}(1 - F(\theta_i)) \geq 0$. Hence, $\mathbb{E}\Pi^*$ and $R_{n+1}^*$ are both non-decreasing in $r$. In particular they are increasing for $r \in \left(0, q_1\theta_{n+1}^*\right]$ then constant in $r$ for $r > q_1\theta_{n+1}^*$. □

*Proof of Proposition 4.5.6.* By Proposition 4.5.4, $\theta_{n+1}^*$ is unique and such that $g(\theta_{n+1}^*) = 0$. Also, we can write:

$$\left.\frac{\partial \mathbb{E}\Pi(R_{n+1})}{\partial r}\right|_{R_{n+1}=R_{n+1}^*} = \begin{cases} 0 & \text{if } r_b^* \leq t_1 \\ \sum_{i=1}^{j}\left[1 - F(\theta_i) - \theta_i f(\theta_i)\right] & \\ & \text{if } t_j < r_b^* \leq \min\{t_{j+1}, \bar{t}\}, j = 1, ..., m \\ \sum_{i=1}^{n}\left[1 - F(\theta_i) - \theta_i f(\theta_i)\right] & \text{if } \bar{t} < r_b^* \end{cases}$$

Since $\mathbb{E}\Pi$ is concave in $\theta_{n+1}$, we have that $g(\theta) \geq 0$ for $\theta < \theta_{n+1}^*$. Hence, $\sum_{i=1}^{j}\left[1 - F(\theta_i) - \theta_i f(\theta_i)\right] > 0$ for $\theta < \theta_{n+1}^*$ such that $\left.\frac{\partial \mathbb{E}\Pi(R_{n+1})}{\partial r}\right|_{R_{n+1}=R_{n+1}^*} > 0$. It is therefore optimal to set $r$ so high that no customer buys individual components, i.e., pure bundling is the best strategy. □

*Proof of Proposition 4.5.7.* Suppose such a mixed bundling strategy is optimal for a particular distribution and the maximum is achieved at $\theta_{b2}^*$ and $r^*$. Let $g(\theta) = \frac{\partial \mathbb{E}\Pi}{\partial R_{n+1}} = 1 - F(\theta) - \theta f(\theta)$, then $g(\underline{\theta}) > 0$ and $g(\overline{\theta}) < 0$. we have the following two cases.

- Case 1: $\theta^*_{b2}$ is the single local maximum point, then $g(\theta^*_{b2}) \geq 0$ and $g(\theta) \geq 0$ for $\theta \leq \theta^*_{b2}$, so $\mathbb{E}\Pi$ increases in $r$ and the pure bundling strategy is optimal, contradicting with optimal mixed bundling strategy.

- Case 2: $\theta^*_{b2}$ is one of the local maximum points. $\theta^*_{b2}$ cannot be the first local maximum point due to the reason shown in Case 1. Suppose $\theta^*_{b2}$ is the second local maximum point. Let $\theta^*_{b1}$ be the first local maximum point. Then, $\theta^*_{b1}$ is the local maximum point for pure bundling strategy.

  We first show that $r^* = \theta^*_{b1} q_1$. Since there are two local maximum points, $g(\theta) < 0$ for $\theta^*_{b1} < \theta < \theta'$ and $g(\theta) > 0$ for $\theta > \theta'$, where $\theta' \leq \theta^*_{b2}$.

  - Case 1: $r^*$ cannot be in the interval $(0, \theta^*_{b1} q_1)$ because $\mathbb{E}\Pi$ increases in $r$ for $0 \leq r < \theta^*_{b1} q_1$.

  - Case 2: $r^*$ cannot be in the interval $(\theta^*_{b1} q_1, \theta' q_1)$ because $\mathbb{E}\Pi$ decrease in $r$ for $\theta^*_{b1} q_1 < r < \theta' q_1$.

  - Case 3: $r^*$ cannot be in the interval $[\theta' q_1, \theta^*_{b2} q_1]$. If $r^*$ is in this interval, then $\mathbb{E}\Pi$ increases in $r$ for $\theta' q_1 \leq r \leq \theta^*_{b2} q_1$ and $r^* = \theta^*_{b2} q_1$ which means pure bundling strategy is optimal.

  Therefore, $r^* = \theta^*_{b1} q_1$. We next show if $\theta^*_{b2}$ achieves the maximum, we get contradiction. Let $R^*_{b1}$ and $R^*_{b2}$ denote the optimal bundle price corresponding to $\theta^*_{b1}$ and $\theta^*_{b2}$ respectively.

$$\begin{aligned}
\mathbb{E}\Pi(R^*_{b2}, r^*) &= (1 - F(\theta^*_{b2}))(Q_{n+1} - Q_2)\theta^*_{b2} + [1 - F(\theta^*_{b1})]\, r^* \\
\mathbb{E}\Pi(R^*_{b1}) &= (1 - F(\theta^*_{b1}))(Q_{n+1})\theta^*_{b1} \\
\mathbb{E}\Pi(R^*_{b2}, r^*) &> \mathbb{E}\Pi(R^*_{b1})
\end{aligned}$$

So,

$$\frac{\theta^*_{b1}}{\theta^*_{b2}} < \frac{1 - F(\theta^*_{b2})}{1 - F(\theta^*_{b1})} \tag{C.6}$$

Also, $\theta^*_{b2}$ satisfies

$$\begin{aligned}
\mathbb{E}\Pi(R^*_{b2}, r^*) &= (1 - F(\theta^*_{b2}))(Q_{n+1} - Q_1)\theta^*_{b2} + [1 - F(\theta^*_{b1})]\, r^* \\
&> (1 - F(\theta^*_{b2}))Q_{n+1}\theta^*_{b2}
\end{aligned}$$

So,

$$\frac{\theta^*_{b1}}{\theta^*_{b2}} > \frac{1 - F(\theta^*_{b2})}{1 - F(\theta^*_{b1})} \tag{C.7}$$

Condition C.7 contradicts with condition C.6; therefore, this mixed bundling strategy cannot be optimal.

Using the similar reasoning, we can prove that if $\theta^*_{b2}$ is any other local maximum point, we get contradictions too. $\square$

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This dissertation was typeset with LaTeX[†] by the author.

[†]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.